Reinventing the wheel: Hodographic solutions to the Kepler problems[a)]

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There are two Kepler problems: given the inverse-square law, find the trajectories; or, given Kepler's laws, find the inverse-square law. Traditionally these problems are solved in the classroom via calculus, but the amount of calculus needed may be prohibitively high for a first-year course. Alternative solutions to the Kepler problems have been discovered, forgotten, and rediscovered for centuries. Many of these employ Hamilton's hodograph, a graphical representation of an object's velocity. This article demonstrates hodographic solutions to the Kepler problems, including an algorithm for the construction of parabolic trajectories.
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I. INTRODUCTION

One of the triumphs of classical mechanics is the demonstration that the observed elliptical orbits of the planets follow directly from Newton's laws of motion and the inverse-square law of gravity. Textbook treatments of ''the Kepler problem'' often begin with the law of gravity and the conservation of angular momentum, develop a differential equation for $u=1/r$ as a function of angle θ, and solve it.[1] The resulting solution is revealed to be the standard polar equation for the family of conic sections. Less commonly the procedure is reversed: Starting from the observed elliptical path of a planet and Kepler's second law (equal areas in equal times), one may derive the inverse square law by differentiation of the solution.[2]

Determining the (unknown) force law from Kepler's empirical laws was the first task facing the natural philosophers of Newton's time, although today the emphasis is reversed. Historians of science usually refer to the determination of the force from Kepler's laws as the ''direct'' Kepler problem. The derivation of the elliptical planetary orbits from Newton's law of gravitation is often called the ''inverse'' Kepler problem.[3]

The standard treatments of both Kepler problems—the force problem and the trajectory problem—require a comfortable facility with the differential calculus of vector-valued functions, and at least a nodding familiarity with the polar equation of the conic sections. An instructor teaching first-year physics, even if calculus-based, often treats only the special and far easier case of circular orbits, because the necessary calculus is a little beyond the reach of almost all of the students. What is wanted is an accessible approach to the solution of the Kepler problems. Such was found nearly two centuries ago, the ''hodograph'' of Sir William Rowan Hamilton and August F. Möbius.[4] This approach has been forgotten and independently rediscovered several times, as often recounted in this journal.[5] Perhaps the best known of these rediscoveries is Feynman's ''Lost Lecture,'' beautifully recreated by D. and J. Goodstein.[6] Feynman credits Fano's treatment of Rutherford scattering[7] for providing the basis of his Kepler solution.[8]

Newton's original solution of the direct problem,[9] while not exactly calculus-based, does not employ the hodograph and will not be discussed here. Newton's discussion of the inverse problem, which does require calculus, presents greater problems. Twenty years ago, Weinstock[10] reopened an old controversy, begun by Johann Bernoulli,[11] arguing that Newton's sketch of a solution to the inverse problem was incorrect. While most historians and mathematicians (notably Arnol'd[12]) give the great Newton the benefit of the doubt, at least one prominent historian of science[13] agrees that Bernoulli, not Newton, should have won Wren's famous prize of a forty shilling book, offered for the solution of the inverse Kepler problem. *Latera non eligo*: The controversy is too fierce and the mysteries too deep for this author to choose sides here. Comprehensive discussions of Newton's work on the Kepler problems have been given by Aiton,[14] Brackenridge,[15] Chandrasekhar,[16] De Gandt,[17] Densmore,[18] and Whiteside,[19] to name but a few.

This article first states a few theorems for the ellipse and the parabola. Next, it introduces Hamilton's hodograph, with some history. The hodograph is used to construct the familiar trajectory of projectile motion. Two key theorems for the Kepler problems are sketched. Finally, the hodographic solutions to both Kepler problems are given. For the inverse problem, the hodograph provides both the familiar elliptic and a new parabolic trajectory.

II. SOME GEOMETRY OF THE ELLIPSE AND THE PARABOLA

The difficulty with teaching geometric solutions is that students of the current generation do not know much about the conic sections. Fortunately the number of theorems needed is small. Classical geometric proofs of most of these theorems may be found in the *Conics* of Apollonius of Perga, written twenty-two centuries ago.[20] Exceptions are those proofs dealing with the focus of a parabola and the directrix, neither of which appears in the *Conics*. Analytic proofs are usually not difficult to construct. To keep this article to a manageable length, the conic theorems are simply stated.

A. The ellipse

1. The perpendiculars from the tangent to the foci

Given a tangent, draw two lines of length r_1, r_2 from the foci perpendicular to the tangent. Then $r_1 r_2 = b^2$, where b is the semiminor axis of the ellipse. (This theorem is not in the *Conics*, but it follows as a corollary to III. 46.) In Fig. 1, ZM and YS are perpendicular to DE. Then $\mathrm{ZM}\cdot\mathrm{YS}=b^2$.

2. The optical theorem for ellipses

Given an ellipse and a tangent to the ellipse, lines drawn from the foci to the tangent make congruent angles with the

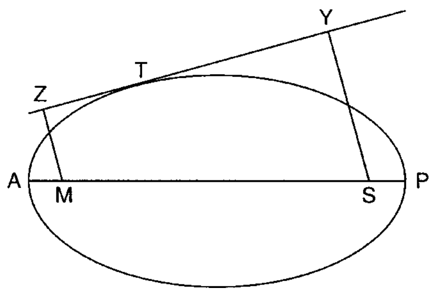

Fig. 1. ZM·YS $= b^2$.

tangent (Fig. 2). Draw lines from M and S to T; then ∠DTM=∠ETS. The theorem is famous but few know Apollonius' ingenious and very beautiful proof (*Conics* III 48).

3. The director circle and the auxiliary circle

Consider an ellipse with center C and foci M and S as shown (Fig. 3), with semimajor axis a and semiminor axis b. Let ZT be a tangent to the ellipse, and as before, draw ZM and YS perpendicular to ZT. Let U be the reflection of M through Z, and thus MZ=UZ. Then the locus of U is a circle of radius $2a$, centered on S, called the *director circle*.[21] (There is a second director circle, congruent to the first, centered on the other focus, M.)

The director circle is the main idea in Feynman's solution to the inverse problem. Start with a circle centered on S. Choose a second point M inside the circle, and draw lines from both S and M to a common point U on the circle. Draw the perpendicular bisector ZY of MU. Call the intersection of ZY with SU a point T. The locus of T is an ellipse. The point M will become one focus of the constructed ellipse; the circle's center S will be the other focus.

Now consider the locus of Z. A line drawn from the center C to Z divides the triangle ΔMUS proportionally, as MC $=\frac{1}{2}$MS and MZ$=\frac{1}{2}$MU. Therefore CZ is parallel to US and its length is half the fixed length of US: the locus of Z is also a circle, of radius a, centered on C. This circle, which circumscribes the ellipse, is called the *auxiliary circle*.[22] The proof that the locus of Z (and Y) is a circle is given in *Conics*, III 50. The auxiliary circle is used by Kelvin and Tait to prove Hamilton's theorem (see below).

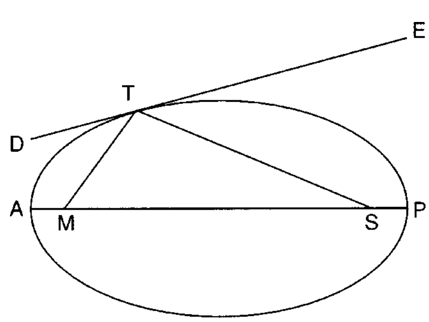

Fig. 2. Ellipse optical theorem: ∠DTM=∠ETS.

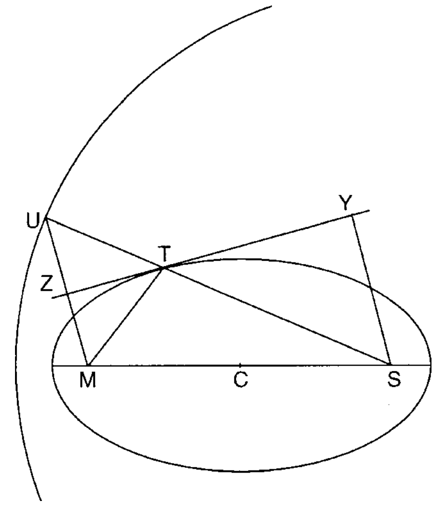

Fig. 3. Director circle for the ellipse.

B. The parabola

1. The optical theorem for parabolas

Everyone knows that a ray of light striking a parabolic mirror parallel to the axis reflects to the focus (Fig. 4). Given a parabola AT with vertex A, focus M, and axis MA. let a ray LT be drawn parallel to the axis. If LT is reflected about the normal at T, it will strike the focus at M. That is, ∠DTM=∠ETL.[23]

2. Newton's theorem on parabola tangents

We will need a theorem proved by Newton himself: The foot of a perpendicular line drawn from the focus to a tangent lies on the tangent at the vertex. (*Principia*, Corollary III to Lemma XIV, following Proposition XII) (Fig. 5).

Draw the line from focus S perpendicular to the tangent TP. Let Z be the foot of this perpendicular line on TP, and denote the vertex by A. Then ZA is the tangent at the vertex, and Z lies on this tangent.

3. The ''director circle'' and the ''auxiliary circle'' for the parabola

Recall the definition of the director circle for the ellipse, and follow the same prescription for the parabola (Fig. 6). Let SZ be the perpendicular line drawn from S to the tangent PT. Then U, a point that should be on the director circle, is the reflection of S through Z. For the parabola, though, the locus of U is easily seen to be a straight line. This follows from Newton's theorem: The locus of Z is the tangent at the vertex. Draw a line from U parallel to the axis AS to intersect the vertex tangent AZ at Y. Since UZ=ZS, and ∠UZY=∠AZS, the distance UY will always equal the dis-

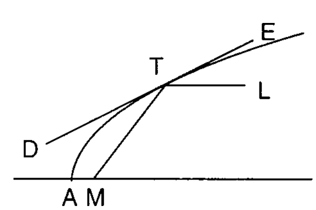

Fig. 4. Parabola optical theorem: ∠DTM=∠ETL.

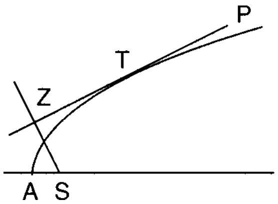

Fig. 5. Newton's theorem: ZA is the tangent at the vertex.

tance AS, which is fixed. Then the locus of U is a vertical line, just as far from the vertex as is the focus; it's the directrix! Recall that a parabola may be thought of as an ellipse with one focus at infinity; so the radius of the director circle becomes infinite, i.e., the director circle becomes a straight line. In the same way, the auxiliary circle, the locus of Z, becomes the tangent at the vertex.[24]

III. HAMILTON'S HODOGRAPH

Hamilton evidently became interested in planetary physics following the discovery of the planet Neptune in 1846.[25] While preparing a series of lectures on this subject, he discovered a new way of thinking about orbits, with a graphical representation of the velocity. Hamilton called the graph of the tip of the velocity vector as a function of time the *hodograph*, from the Greek ''to draw'' (*graphein*) and ''path'' (*hodos*). The arc of the hodograph between two times infinitesimally close to each other is proportional to, and in the same direction as, the acceleration. That is, the ''velocity'' of the hodograph is the acceleration of the object.

Hamilton communicated his results to the main British participants (G. B. Airy, J. C. Adams, W. Whewell) in the search for the planet Neptune, as well as to William Thomson (later known as Lord Kelvin). In general the hodograph was not received enthusiastically, except by the ''Celtic school'' of mathematical physicists: Thomson, his close collaborator Peter Guthrie Tait, and James Clerk Maxwell, as well as Hamilton himself, all of whom would use Hamilton's device in their textbooks. In Hamilton's case the hodograph crops up in various works on quaternions. As it happens, one of the only American texts to describe the hodograph at all was Josiah Williard Gibbs's on vector analysis,[26] who perhaps wished to show that his rival methods were easily equal to the job of describing the hodograph, linked so closely to the father of quaternions. The hodograph does not make any notable appearance in an American textbook until its rediscoveries by Fano[27] and Feynman.[28] It may be that the hodograph was buried along with quaternions.

Given the hodograph and its origin, integration can determine the position graph. However, the position can also be found geometrically, with a construction. There are two separate problems arising in the construction. First, given a

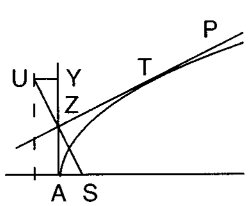

Fig. 6. ''Director circle'' for the parabola.

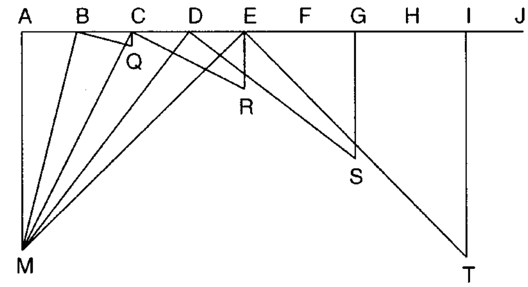

Fig. 7. Constructing a projectile's trajectory from its hodograph.

velocity, at what point in the plane is it to be the tangent? We do not know where to place it, even if we know its orientation and its size. Second we will be constructing a position graph atop a velocity graph; somehow we must relate the scales of position and velocity. Rotating the hodograph through 90° counterclockwise solves both problems. We choose the initial position, so that is where we draw the first tangent. We know its direction: perpendicular to the first ray of the hodograph. The next and successive positions will be determined, as will be shown, by the relative scale between the position and velocity graphs. Angular momentum provides an easy method for setting this scale. By rotating the velocity, we will obtain relationships involving products of the ratios $v_y : v_x$ and $x:y$, which lead naturally to constraints involving the angular momentum. These constraints will provide the scale.

It is only right that angular momentum should provide the key to the hodograph construction, as it was Kepler himself who first understood the importance of angular momentum, and who provided its first expression in his second law.

Let's see how this works with a simple example. Throw an object horizontally. Its horizontal velocity is constant. Its vertical velocity begins with a value of zero and increases at a constant rate. This projectile's hodograph is a vertical line. To construct its trajectory from its hodograph, begin by rotating the hodograph to a horizontal line, denoted AJ in Fig. 7. The original position of the projectile is point A, and MA is the original velocity; MB is the velocity after an interval Δt; MC is the velocity after an interval of $2\Delta t, \ldots$. At each velocity's tip A, B, C,... draw perpendiculars to the velocities MA, MB, MC,... . These perpendiculars show the actual directions of the velocities (and hence the tangents). For each velocity's tip, e.g., ME's, drop a perpendicular line from the hodograph at twice the distance of that tip from A (for E, the corresponding point is I). The intersection of this line with the line drawn perpendicular to the corresponding velocity (here, point T) lies on the trajectory.

To show the constructed curve is a parabola, draw a line ZY parallel to the hodograph AG, where ZA=AM (Fig. 8). Draw a line from G perpendicular to AG, intersecting ZY at V, and a line from D to V. Then $\angle$MDA=$\angle$VDG and MD=DV. By construction, DS is perpendicular to MD and thus to DV, so point S lies equidistant from both the line ZY, now revealed to be a directrix, and the hodograph origin, M, the focus.

Identify this parabola as the projectile's actual trajectory with the help of angular momentum (Fig. 9). The line DW is to be tangent at some point W with coordinates $(\beta x, \beta y)$ on the position graph, where β is some scale factor to be determined. Point W is the intersection of a perpendicular line

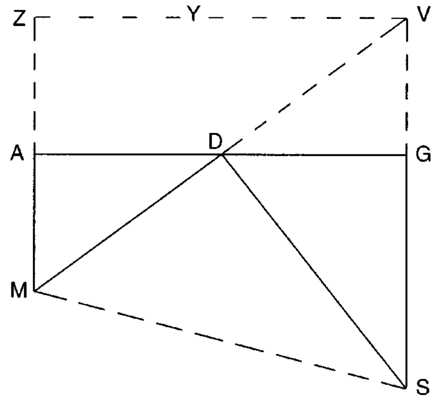

Fig. 8. The construction is a parabola.

dropped from some point, K on the hodograph. If the parabola constructed is the actual trajectory, we must find a physical justification for the claim AK=2AD, or what is the same thing, $\beta x = 2v_y$.

The original horizontal velocity v_x equals the original velocity, v_0, which is just MA. Later the velocity MD has components $(v_0, v_y) = (\text{MA},\text{AD})$. By similar triangles,

$$v_y : v_0 = \beta y : (\beta x - v_y)$$

or

$$\beta(xv_y - yv_0) = v_y^2. \tag{3.1}$$

The quantity on the left is just $\beta(L/m)$, where L is the size of the angular momentum, and m the mass of the projectile. By conservation of energy, $v_y^2 = 2gy$, so we can say

$$\beta L = 2mgy. \tag{3.2}$$

But $|d\mathbf{L}/dt| = dL/dt = |\mathbf{r}\times\mathbf{F}| = mgx$. Differentiating the above equation,

$$\beta mgx = 2mgv_y. \tag{3.3}$$

Consequently $\beta x = 2v_y$ so AK=2AD, and the physical path is the same as the constructed parabola. For completeness, note $x = v_0 t$, and $v_y = gt$, so $\beta = 2g/v_0$. Below, a related procedure will produce conic sections for the inverse Kepler problem.

To solve the Kepler problems with the hodograph, we need only two theorems, due to Hamilton:

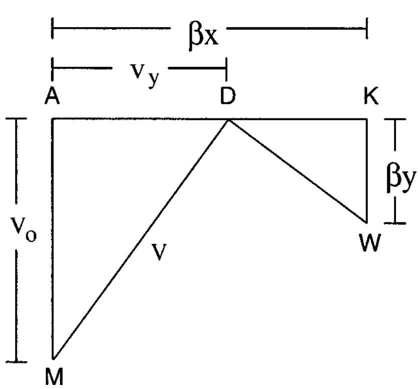

Fig. 9. Physically justifying the trajectory.

(i) If a mass moves under the action of an inverse square law force toward a fixed point, then the hodograph is a circle whose center is the fixed point.

(ii) If Kepler's second law holds, and if the trajectory of a mass is a conic section, then the hodograph is a circle. (Kelvin and Tait call this ''Hamilton's Theorem.''[29])

Goldstein has given a very quick proof of (i) based on the constancy of the Runge–Lenz vector.[30] Kelvin and Tait, in their influential mechanics text *A Treatise on Natural Philosophy*, provided an elegant analytic proof,[31] reprinted in Hankin's valuable biography of Hamilton.[32] Nearly identical geometric proofs are given by Hamilton,[33] Kelvin and Tait,[34] Maxwell,[35] Fano,[36] and Feynman,[37] as follows. Let the force be attractive (but this is not necessary). Then an inverse square law means ($\hat{\mathbf{r}}$ is the unit vector from the force center to the mass)

$$\mathbf{a} = -C/r^2\hat{\mathbf{r}} = \Delta\mathbf{v}/\Delta t = (\Delta v/\Delta\theta)(\Delta\theta/\Delta t). \tag{3.4}$$

It is easy to show geometrically that if the net force on an object is directed radially, then Kepler's second law follows and angular momentum is conserved.[38] Assuming this,

$$mr^2\Delta\theta/\Delta t = L, \text{ a constant, or } \Delta\theta/\Delta t = L/mr^2. \tag{3.5}$$

Putting these together gives

$$(\Delta\mathbf{v}/\Delta\theta)(L/mr^2) = -C/r^2\hat{\mathbf{r}}$$

or

$$\Delta\mathbf{v} = -K\hat{\mathbf{r}}\Delta\theta. \tag{3.6}$$

The size of the change in velocity is proportional to the change in angle, and its direction is determined by the radius vector. If we imagine dividing a trajectory into sectors of equal angles $\Delta\theta$, then the sum of all the $\Delta\mathbf{v}$'s will make a regular polygon, because the successive changes in the velocity vectors are all inclined to each other at the same angle, $\Delta\theta$, and all the $\Delta\mathbf{v}$'s will be the same size. $K\Delta\theta$. In the limit as $\Delta\theta\to 0$, the regular polygon becomes a circle of radius K. (For the Kepler problem, $C = GM$, so $K = GMm/L$, a result first given by Hamilton.[39]) The center of the circle is just the point from which $\hat{\mathbf{r}}$ is drawn. To see this, recall that the ''velocity'' of the hodograph, tangent to the circle, is really the acceleration, **a**. If the hodograph is rotated 90°, it remains a circle, but now all the hodograph rays are perpendicular to the actual directions of the velocities. That means that the direction of the actual accelerations will be perpendicular to the tangents of the rotated hodograph, i.e., they will be radial. We know that the central force provides radial acceleration. That is, for the Kepler problem, the rotation of the hodograph by 90° not only will provide the scale of the trajectories (as we will see shortly), it also assures us that the center of the hodograph is just the center of the force. That proves (i). It is worthwhile to see this procedure worked out in detail.

In Fig. 10, an orbit is divided into eight equal angles of 45° each. The displacements are sketched from A to B, B to C, and so on. These directions give the direction of the various v's, but the size of the displacements are not all in the same proportion as the average velocities, as different intervals take different times. If the common size of the $\Delta\mathbf{v}$'s is set arbitrarily, the sizes of the other velocities are determined, because the directions are set by the original diagram. The normals to the rotated $\Delta\mathbf{v}$'s are all directed toward a

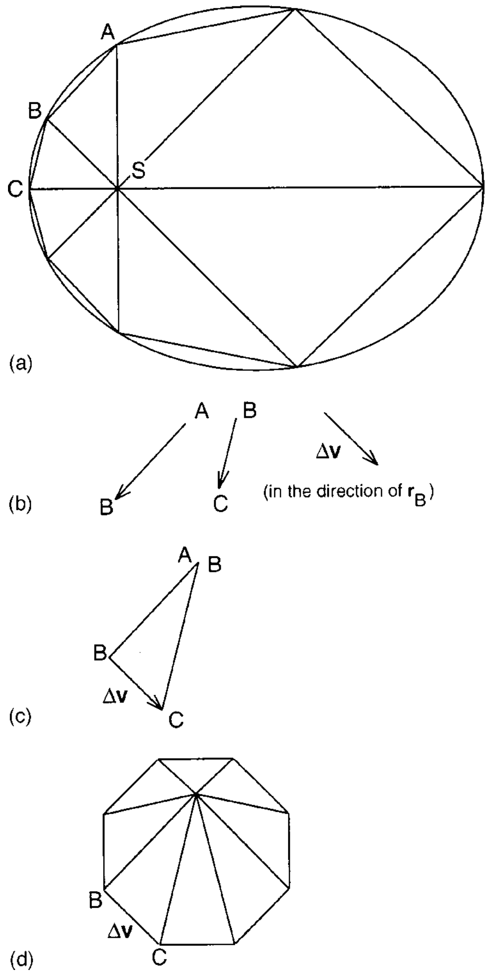

Fig. 10. Construction of the hodograph for an orbit: (a) the orbit divided into 8 arcs of 45 degrees, (b) pulling out the velocities, choosing a size for Δv, (c) subtracting the velocities (resizing AB, BC), (d) the completed hodograph: an octagon.

fixed point. Draw all eight average velocities from this common fixed point, and connect their heads by the appropriate $\Delta\mathbf{v}$. What results is an octagon. Choosing a smaller angle $\Delta\theta$ increases the number of sides, and the polygon hodograph more nearly approximates a circle. Note that the *origin* of the hodograph, the common end point of all the velocity vectors, is *not* the center of the circle; these two points will turn out to be the foci of elliptical or hyperbolic orbits. For the parabola, the circle's center will be the single focus of the curve.

The proof of (ii) is more complicated (for uniform circular motion, however, it is obvious). We should consider separately the three cases of hyperbola, ellipse, and parabola. Proofs for the ellipse and the hyperbola are nearly identical, so we'll consider here only those for the ellipse and the parabola. Maxwell dealt only with elliptical orbits, and based his proof on the director circle.[40] Kelvin and Tait treated all three conics, but used the auxiliary circle (the tangent at the vertex for the parabola.).[41] Maxwell's proof seems simpler and more physical, and it can be extended to the parabola without difficulty; that extension is given below. First, Maxwell's demonstration for the ellipse is given (Fig. 11).

Consider the ellipse ATP with foci M, S (S standing for the sun, A the aphelion, and P the perihelion). Let T be any point on the ellipse, and draw SU through T, such that

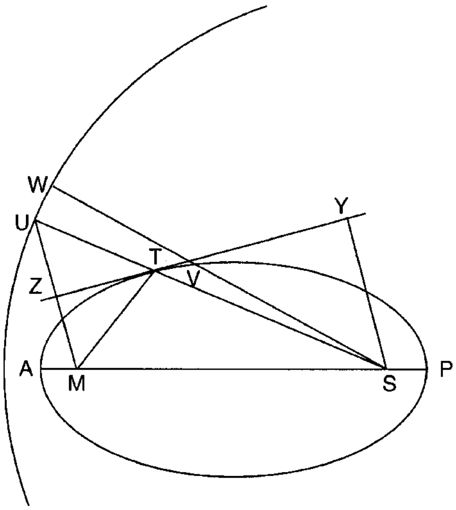

Fig. 11. Finding the hodograph for an elliptical orbit.

SU=AP. Draw a line from M to U. It remains to be shown that UM is perpendicular to, and proportional to, the velocity at point T, and that the locus of U is a circle.

($E1$) *In the ellipse, UM is perpendicular to the velocity at T.* Draw a tangent from T to intersect UM at Z. Then by the ellipse optical theorem, ∠STY=∠MTZ, so ∠MTZ =∠UTZ. Also, UT=SU−TS=AP−TS=MT, so ZT⊥UM. Then the direction of UM is perpendicular to the tangent, and hence the velocity, at T.

($E2$) *In the ellipse, UM is proportional to the velocity at T.* Draw a perpendicular line from S to the tangent to intersect the tangent at Y. Let $\mathbf{v}$ be the velocity at T, of length v. By the conservation of angular momentum, $v\cdot \mathrm{YS}=h$, a constant. Recall (from Sec. II A 1 above) $\mathrm{ZM}\cdot\mathrm{YS}=b^2$. Then

$$1/\mathrm{YS}=v/h=\mathrm{ZM}/b^2,$$

or

$$\mathrm{ZM}=(b^2/h)v=\tfrac{1}{2}\mathrm{UM} \tag{3.7}$$

so UM is proportional to v.

Since SU is always equal to the major axis, it follows that the locus of U is a circle, with common origin of the velocity vectors at M; this circle is the hodograph turned through 90° (because UM is perpendicular to $\mathbf{v}$). For comparison with the parabolic case, note that we can write

$$\mathrm{UM}=(2ap/h)v, \tag{3.8}$$

where $p=b^2/a$ is the "parameter" or *latus rectum* of the ellipse; the perpendicular distance from the axis at the focus to the ellipse.

Now for the parabola (Fig. 12). Consider the parabola AP, focus at S, axis AS, vertex A, directrix LD; let $\mathrm{AD}=a$, so $\mathrm{AS}=a$ and $\mathrm{DS}=2a$. Let T be any point on the parabola. Draw the tangent at T and draw KT parallel to AS. Extend the axis from S to M so that SM=DS. Draw a line from S through T to point U so that SU=SM=DS. Finally, draw MU. As before, the locus of U is a circle, as $\mathrm{SU}=2a$, a fixed length. It remains to be shown that UM is perpendicular to, and proportional to, the velocity at point T.

($P1$) *In the parabola, UM is perpendicular to the velocity at T.* Draw the tangent at the vertex, and let it intersect at Z the tangent drawn at T. Draw the line SZ, which by New-

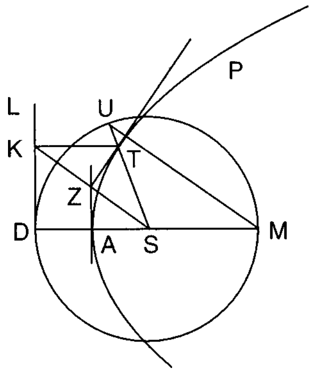

Fig. 12. Finding the hodograph for a parabolic orbit.

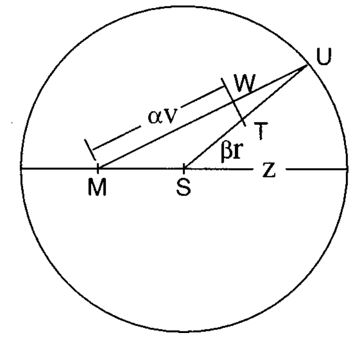

Fig. 13. Finding the right distance.

ton's theorem is perpendicular to the tangent TZ. Draw ZK. By the optical theorem, $\angle$KTZ$=\angle$STZ. T, on the parabola, is equidistant from LD and from S, so ΔSTZ$=\Delta$KTZ, and S, Z, and K are collinear. Then $\angle$TKZ$=\angle$TSZ, but $\angle$TKZ$=\angle$ZSA. Then $\angle$TSA is twice as large as $\angle$ZSA, and hence $\angle$UMS$=\angle$ZSA. Then UM is parallel to SZ, and as SZ is perpendicular to the tangent, so too is UM.

($P2$) *In the parabola, UM is proportional to the velocity at T.* Since the rate at which the area changes is constant, $v\cdot \text{SZ}=h$, a constant, so $(v/h)=1/\text{SZ}$. From ($P1$), UM is parallel to SK. Then SK:UM=ST:US=2SZ:UM, so

$$\text{ST}=2\,\text{SZ}(\text{US}/\text{UM}). \tag{3.9}$$

We can also say ΔTZK$=\Delta$TZS and $\angle$ZSA$=\angle$ZKT, so

$$\text{ST}=\text{SZ}^2/\text{AS}. \tag{3.10}$$

Setting these two expressions equal to each other, we find (recall AS$=a$, and US$=2a$)

$$\text{UM}=4a^2/\text{SZ}=4(a^2/h)v \tag{3.11}$$

so UM is again proportional to v. The *latus rectum* for a parabola, defined exactly the same as for an ellipse, has the value $2a$. Again we can write

$$\text{UM}=(2ap/h)v, \tag{3.8$'$}$$

in agreement with the earlier expression (3.8) for the ellipse.

Given Kepler's second law, and either an elliptical or parabolic trajectory, the hodograph has been shown to be a circle. The same can be done for hyperbolic trajectories as well. Therefore, given Kepler's second law and a conic trajectory, the hodograph is a circle. That proves Hamilton's Theorem, (ii).

IV. THE HODOGRAPHIC SOLUTIONS

A. The direct problem

We want to show that, given Kepler's Laws, the force is (i) directed toward one focus, and (ii) inverse square. We follow Maxwell's proof[42] (a different but similar version is given by Kelvin and Tait[43]).

Kepler's first law says that the trajectory is an ellipse. Hamilton's theorem tells us that Kepler's first two laws guarantee the hodograph is a circle. Let the hodograph be as shown in Fig. 10, a circle with center S and radius equal to AP. The tip of the velocity at U will move along the arc (toward W, say) of the circle. As the direction of the velocity is perpendicular to UM, so the acceleration is perpendicular to the tangent of the circle; that is, it is directed toward S. The force is parallel to the acceleration, and so it too is directed toward S, a focus of the ellipse. That proves (i). (A similar proof could be constructed for parabolic trajectories.)

By Kepler's second law, $r^2\Delta\theta/\Delta t=h$, a constant, so $\Delta\theta/\Delta t\propto 1/r^2$. In a time Δt let the hodograph tip move from U to W. Because the hodograph is a circle, $\Delta v\propto\Delta\theta$ between neighboring points T and V on the conic. That is, the acceleration $a=\Delta v/\Delta t\propto\Delta\theta/\Delta t$, and therefore $a\propto 1/r^2$. That proves (ii), and the direct problem is solved.

B. The inverse problem

Now we want to construct, geometrically, the trajectories corresponding to the hodograph. With an inverse square force directed toward the sun, the hodograph is a circle. Let the hodograph have a radius z, let its origin M be inside the circle, and let its center be S, corresponding to the sun (Fig. 13). We know that the center of the hodograph corresponds to the sun, because (having rotated the hodograph) the accelerations are radial. For a given velocity $v=$MU, draw a perpendicular line at W a distance αv from the origin M where α is some numerical factor. If α is chosen correctly, a point on the trajectory will be found by the intersection T of this perpendicular line (which is the actual direction of the tangent) with the radius drawn to U; since the accelerations are radial, the radius vector ST corresponding to MU must lie on the line SU. *Any* perpendicular line to the rotated hodograph ray will be parallel to the actual velocity, and therefore will be in the right direction for a tangent; the envelope of all tangents will give us the trajectory. Only one question remains: How far along MU should W be chosen, i.e., what is the value of α?

From what we know of the director circle, we might think that W should be the bisector of MU as shown in Fig. 13. This is Feynman's construction. In fact, *no other choice* of a perpendicular line to UM will produce an ellipse, as a little trial and error will show. Feynman, in the short span of the allotted hour for his ''Lost Lecture,'' did not explain why the unique *geometric* choice of the bisector, or $\alpha=\frac{1}{2}$, is the unique *physical* choice. The answer, as in the horizontal projectile, again depends on angular momentum. There, the answer followed from the change of L; here, from the need to conserve it.

Consider the apsides (Fig. 14). For the aphelion velocity $v_a=$MD, the perpendicular will be at point A, where

$$\text{AM}=\alpha v_a \tag{4.1}$$

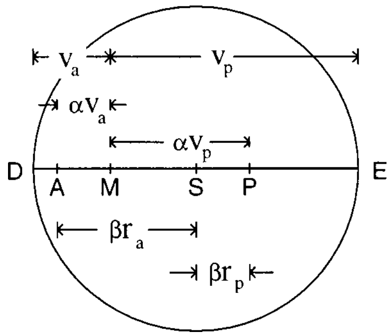

Fig. 14. Setting the scale for the ellipse.

and similarly for the perihelion velocity $v_p = \mathrm{ME}$, the perpendicular will be at point P, where

$$\mathrm{MP} = \alpha v_p . \tag{4.2}$$

Then the radii r_p and r_a for the perihelion and aphelion, respectively, are found from the relations (again, β is some scale factor)

$$\beta r_p = \mathrm{SP} = \mathrm{MP} - \mathrm{MS} = \alpha v_p - \mathrm{MS}, \tag{4.3}$$

$$\beta r_a = \mathrm{SA} = \mathrm{MS} + \mathrm{AM} = \mathrm{MS} + \alpha v_a . \tag{4.4}$$

Conservation of angular momentum at the apsides demands $L = m r_a v_a = m r_p v_p$. Substituting in the expressions for the perihelion and aphelion radii, and rearranging, leads to

$$\alpha (v_p^2 - v_a^2) = \mathrm{MS}(v_p + v_a) \tag{4.5}$$

or dividing,

$$\alpha (v_p - v_a) = \mathrm{MS}. \tag{4.6}$$

But

$$v_p = z + \mathrm{MS}, \quad v_a = z - \mathrm{MS} \tag{4.7}$$

so $\alpha = \frac{1}{2}$. Consequently, if the hodograph origin lies inside the circle, we are forced by angular momentum conservation into the construction of an elliptical orbit.

This construction conserves angular momentum at the apsides. Is L conserved everywhere? Let the radius of the hodograph equal z, as before, and for convenience let $\mathrm{MS} = k$ (Fig. 15). The orbital radii are obtained by Feynman's construction; draw the perpendicular bisector at J of the velocity UM to intersect SU at T; ST is the radius vector corresponding to the velocity UM, and the locus of T is the elliptical trajectory. Let $\theta = \angle \mathrm{MUS}$. Then by construction $\frac{1}{2} v = (z - \beta r)\cos\theta$, or

$$\beta r \cos\theta = z \cos\theta - \tfrac{1}{2} v . \tag{4.8}$$

The constant value of L may be found by evaluating it for either apside. For the perihelion $L_p = m v_p r_p$; but from the scale of the orbit in the hodograph found earlier,

$$\beta r_p = \tfrac{1}{2} v_p - k = \tfrac{1}{2}(z - k), \tag{4.9}$$

$$L_p = m v_p r_p = \tfrac{1}{2}(m/\beta)(z^2 - k^2). \tag{4.10}$$

The value of the angular momentum at any radius r is, using (4.8) (note the definition of the angle),

$$\begin{aligned} L &= m v r \sin(90 - \theta) \\ &= m v r \cos\theta \\ &= (m/\beta) v z \cos\theta - \tfrac{1}{2}(m/\beta) v^2 . \end{aligned} \tag{4.11}$$

We need to show that this is the same as $\frac{1}{2}(m/\beta)(z^2 - k^2)$. By the law of cosines, $k^2 = v^2 + z^2 - 2vz\cos\theta$. Multiplying both sides of this equation by $\frac{1}{2}(m/\beta)$ gives

$$\begin{aligned} L &= (m/\beta) v z \cos\theta - \tfrac{1}{2}(m/\beta) v^2 \\ &= \tfrac{1}{2}(m/\beta)(z^2 - k^2) = L_p , \end{aligned}$$

so angular momentum is conserved.

For completeness, let's find the value of β. We know that the hodograph is just the director circle; i.e., $z = 2\beta a$. Also $k = 2\beta c$, the distance between the foci. Then

$$\begin{aligned} z^2 - k^2 &= 4\beta^2 (a^2 - c^2) = 4\beta^2 b^2, \\ \beta &= L/2mb^2. \end{aligned} \tag{4.12}$$

It is known for elliptical orbits that $E = -GMm/2a$, but $E = E_p = L^2/2mr_p^2 - GMm/r_p$. Since $r_p = a - c$, we can find $L^2 = GMm^2 b^2/a = GMm^2 p$, where p is the *latus rectum* for an ellipse, $p = b^2/a$. Then

$$\beta = (\tfrac{1}{2})(GM/ab^2)^{1/2} = (1/2a)(GM/p)^{1/2} \tag{4.13}$$

and $z = 2a\beta = GMm/L$, as we found earlier.

For later reference, note that we can also say

$$L = 2map\beta = (GMm^2 p)^{1/2}. \tag{4.14}$$

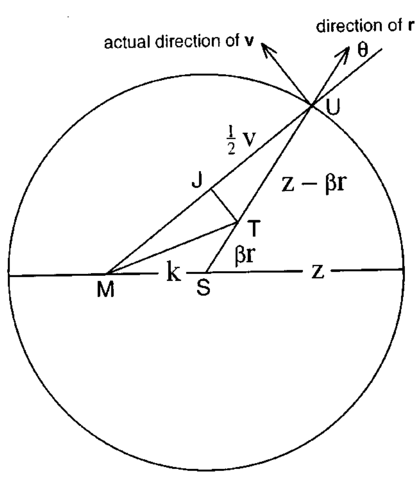

Fig. 15. Conservation of angular momentum for the ellipse.

If the hodograph origin lies outside the circle, the construction above will produce a hyperbolic trajectory, as can be shown very quickly. It might be expected that if the origin lies on the circle, the algorithm would produce a parabola. A few moments with pencil and paper will show that the algorithm fails badly in this case. In what follows, a different algorithm is described and shown to produce a parabolic trajectory which indeed conserves angular momentum.

As before, we have a circular hodograph of radius z with center S and hodograph origin M, on the circle as shown (Fig. 16). Draw the ''director circle''—the directrix, KD—as shown. Following the same procedure as before, draw a chord from M and a radius from the center S to a common point U on the circle. Now draw a line from S parallel to the chord UM to the directrix at W. Finally, draw a line from W

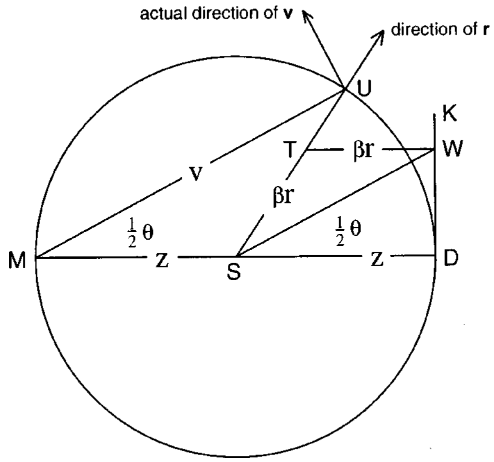

Fig. 16. Conservation of angular momentum for the parabola.

perpendicular to the directrix, intersecting the radius SU at T. ST is the radius vector corresponding to the velocity UM, and the locus of T will be the trajectory.

By construction, the line SW is parallel to UM, so $\angle$UMS$=\angle$WSD. On the other hand, $\angle$USD has twice the measure of $\angle$UMS, because both angles cut off the same arc, but the former angle is inscribed, while the latter is a central angle. Then $\angle$USW$=\angle$WSD$=\angle$TWS since TK is parallel to SD, so T is equidistant from the directrix and the focus S; the locus of T is a parabola.

To show the conservation of angular momentum, let's first find the constant value of L. The greatest velocity the object attains is equal to MD$=2z$. This occurs for the perihelion, at a distance of $\frac{1}{2}(z/\beta)=a$ from D (recall the distance from the focus to the vertex is a). The angular momentum at the perihelion is thus

$$L_p=m(2z)(\tfrac{1}{2}z/\beta)=(m/\beta)z^2. \tag{4.15}$$

At an arbitrary point T, let θ be the central angle between ST and the parabola axis SD, and

$$L=mvr\sin(90-\tfrac{1}{2}\theta)=mvr\cos\tfrac{1}{2}\theta.$$

From the diagram (Fig. 16),

$$z=\beta r+\beta r\cos\theta=2\beta r\cos^2\tfrac{1}{2}\theta, \tag{4.16}$$

$$v\sin\tfrac{1}{2}\theta=z\sin\theta=2z\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta. \tag{4.17}$$

Solving for v and plugging into (4.16) leads to

$$L=2mzr\cos^2\tfrac{1}{2}\theta=(m/\beta)z^2=L_p$$

and once again angular momentum is conserved. It is well known that the total energy of a parabolic orbit is zero. The value of β may be obtained from considering the energy at the perihelion; $E=\frac{1}{2}mv_p^2-GMm/r_p=0$, and using $v_p=2z$, $r_p=a$, and $z=2\beta a$ we find

$$\beta=(GM/8a^3)^{1/2}=(1/2a)(GM/p)^{1/2}; \tag{4.18}$$

for the parabola, the *latus rectum* $p=2a$. Notice that formally β is identical for the ellipse and the parabola, as might have been expected, though the *latus rectum* has different values for the two conics.

We can write the angular momentum for the parabola as

$$L=(m/\beta)z^2=2map\beta=(GMm^2p)^{1/2}, \tag{4.19}$$

which is again formally identical to the ellipse expression. Solving for p and using $z=2\beta a$, we again obtain Hamilton's result for the radius of the hodograph $z=GMm/L$, exactly as before. Thus, an inverse-square law produces a circular hodograph; a circular hodograph generates

- a parabolic trajectory when the hodograph origin lies on the circle,
- an elliptical orbit when the hodograph origin lies inside the circle, and by extension
- a hyperbolic trajectory when the hodograph origin lies outside the circle;

in each case the constancy of angular momentum guarantees that we have found the right scale. Then the inverse-square law leads to a conic section trajectory, and the inverse problem is solved.

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[a]For Peter W. Higgs FRS, my teacher and my friend, in anticipation of the inevitable discovery.

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ROCKET SCIENTISTS

In the financial industry, physicists, along with other scientists, engineers and mathematicians, generally apply their skills to what is called quantitative finance. For that reason, such people earn the nickname ''quants.'' Another sobriquet is ''rocket scientists.'' Frankly, these are not terms of endearment. They are mildly disparaging labels that tend to distract the listener from fully perceiving the value added by physicists. Unfortunately, many physicists willingly comply with this pigeonholing. This is true not only on Wall Street but throughout society. For reasons unfathomable, physicists accept, and perhaps enjoy, being considered rumpled, eccentric and prone to irrational bursts of intellectual energy.

Joseph M. Pimbley, ''Physicists in Finance,'' Physics Today **50** (1), 42–47 (1997).