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Perturbed isotropic harmonic oscillator

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Abstract. The isotropic harmonic oscillator is studied under the action of a central perturbative force $n\beta/r^{n-1}$, where n is a non-zero integer. The case $n=2$ is solved exactly. For other values of n a perturbative method that allows the determination of the speed of precession and the polar equation of the orbit is developed; the method is also applied to the case of the charged isotropic oscillator in a constant magnetic field.

Resumo. O oscilador harmônico isotrópico sob ação de uma força perturbativa de caráter central do tipo $n\beta/r^{n-1}$, onde n é um inteiro diferente de zero, é estudado. O caso $n=2$ é resolvido exatamente. Para outros valores de n um método perturbativo que permite obter a velocidade de precessão e a equação polar da órbita é desenvolvido; o método é aplicado ainda ao caso do oscilador isotrópico carregado em um campo magnético constante.

1. Introduction

In a recent paper published in this journal Sivardière [1] discussed Kepler and harmonic elliptic motion perturbed by an inverse-cube force law. This is an important pedagogical as well as a practical discussion topic since pure Kepler or harmonic motion are never realised in nature. Precession phenomena associated with planetary motion are perhaps the most striking evidence of the importance of this problem. In Sivardière's paper one particular feature in the treatment of the perturbed isotropic harmonic motion has attracted our attention. The author states that if some perturbation is introduced the orbit will be a rosette similar to a precessing ellipse and assumes *a priori* that the polar equation of the perturbed orbit is given by

$$u^2(\theta) = \cos^2(\gamma\theta)/a^2 + \sin^2(\gamma\theta)/b^2. \tag{1}$$

Here $u \equiv 1/r$, γ is a real parameter that measures the departure from a pure elliptical orbit and a and b are respectively the major and minor semi-axes of the ellipse. After substituting (1) into the Binet formula [2] we find that the force $\boldsymbol{F} = F(r)\hat{r}$ acting on a test particle is

$$F(r) = -kr - 2\beta/r^3. \tag{2}$$

Consequently, the associated central potential is

$$U(r) = kr^2/2 - \beta/r^2 \tag{3}$$

where k and β are constants depending on γ. Actually, the solution given by (1) can be obtained by solving the equations of motion directly, a procedure of moderate difficulty that can be taught to the average student at intermediate level.

In further examining the perturbed oscillator problem we have considered the effects of the types of force $n\beta/r^{n+1}$. Some interesting results such as the speed of precession and the form of the orbit can be obtained in an approximate way and compared with results obtained elsewhere [3, 4]. It is our purpose in this paper to discuss the above-mentioned results. We have also included the case of the charged oscillator in a constant magnetic field that can be treated in a similar way.

2. The perturbed isotropic harmonic oscillator

We shall start by recalling briefly the isotropic harmonic oscillator in polar coordinates r and θ. For this case the equations of motion read

$$m\ddot{r} - mr\dot{\theta}^2 + kr = 0 \tag{4}$$

and

$$mr^2\dot{\theta} = l \tag{5}$$

where l is the angular momentum which is conserved due to the central character of the force. Combining (4) and (5) we can write

$$\ddot{r}+\omega^2 r - l^2/m^2r^3 = 0 \qquad (6)$$

where $\omega^2 = k/m$.

The solution of this equation is given in detail below. Consider now the isotropic oscillator perturbed by an inverse-cube force law. The equation of motion in this case reads

$$m\ddot{r} - mr\dot{\theta}^2 + kr + 2\beta/r^3 = 0 \qquad (7)$$

which, combined with the conservation of angular momentum, gives

$$\ddot{r}+\omega^2 r - l'^2/m^2r^3 = 0 \qquad (8)$$

with the difference that now l' is a constant with dimensions of angular momentum

$$l' = (l^2 - 2\beta m)^{1/2}. \qquad (9)$$

Equation (8) is a non-linear differential equation, but fortunately solvable by an analytical procedure that goes as follows. First we integrate the radial coordinate to obtain

$$\dot{r}^2 + \omega^2 r^2 + l'^2/m^2r^2 = 2E/m \qquad (10)$$

which expresses the conservation of mechanical energy. Using (8) and multiplying by r we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}(r\dot{r}) - \dot{r}^2 = -\omega^2 r^2 + \frac{l'^2}{m^2r^2}. \qquad (11)$$

Substituting (10) into (11) we find a simple harmonic oscillator differential equation for the square of the radial coordinate

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(r^2) + 4\omega^2 r^2 = \frac{4E}{m}. \qquad (12)$$

The solution is

$$r^2(t) = (E/k) + A\cos[2(\omega t + \alpha_0)] \qquad (13)$$

where A and α_0 are constants of integration. We can easily relate A with l' and E by substituting the time derivative of (13) into (10). After recalling that $\omega^2 = k/m$ some simple algebra gives

$$A = (E^2 - \omega^2 l'^2)^{1/2}/k. \qquad (14)$$

Now we calculate the time angular variation $\theta(t)$. From (13) and (5) we have

$$\theta(t) - \theta_0 = \frac{kl}{m}\int \frac{\mathrm{d}t}{E + kA\cos[2(\omega t + \alpha_0)]} \qquad (15)$$

where θ_0 is a constant of integration. This is a standard integral [5]

$$\int \frac{\mathrm{d}x}{a + b\cos x} = \frac{2}{(a^2-b^2)^{1/2}}\tan^{-1}\left(\frac{(a-b)\tan\frac{1}{2}x}{(a^2-b^2)^{1/2}}\right)$$

for $a^2 > b^2$. Putting $x = 2\omega t + 2\alpha_0$, after some algebraic manipulation we finally obtain

$$\tan\left(\frac{l'}{l}(\theta - \theta_0)\right) = \frac{E - kA}{\omega l'}\tan(\omega t + \alpha_0). \qquad (16)$$

The reader can see that the same procedure applies to the non-perturbed case ($\beta = 0$); all that is necessary here is to replace l' by l.

Equations (13) and (16) are the parametric equations of the perturbed orbit. If we eliminate time we will obtain equation (1), which describes the orbit in space. As a matter of fact, (1) corresponds to a special choice of the constants α_0 and θ_0. From now on we will assume that $\theta_0 = 0$ and $\alpha_0 = 0$.

Now, if we take the square of (16) (with $\alpha_0 = 0$, $\theta_0 = 0$) and make use of the trigonometric relation $\tan^2\omega t = \cos^{-2}\omega t - 1$, we can write

$$\cos^2(\omega t) = \frac{(E-kA)^2}{\omega^2 l'^2\tan^2(\gamma\theta) + (E-kA)^2} \qquad (17)$$

where we have identified the ratio l'/l with the real parameter γ mentioned earlier. Since $\cos 2\omega t = 2\cos^2\omega t - 1$, we can combine (13) and (17) to obtain finally

$$u^2 \equiv \frac{1}{r^2} = \frac{\cos^2(\gamma\theta)}{a^2} + \frac{\sin^2(\gamma\theta)}{b^2} \qquad (18)$$

where the semi-axes a and b are given respectively by

$$a^2 = (E/k) + A \qquad (19)$$

and

$$b^2 = (E/k) - A \qquad (20)$$

where A is defined in (14). These expressions for a and b are easily obtained from (13) by setting $t = 0$ and $t = \pi/2\omega$ respectively. It also follows from (19) and (20) that the total mechanical energy is given by

$$E = \tfrac{1}{2}k(a^2 + b^2). \qquad (21)$$

When γ is different from unity precession will arise. Let us then calculate the mean precession rate $\langle\Omega\rangle$. In contrast to the Kepler problem the angle between two successive periapses is π/γ (and not $2\pi/\gamma$). Therefore, we have approximately

$$\langle\Omega\rangle \simeq \omega_\theta(1-\gamma)/\gamma \qquad (22)$$

where $\omega_\theta = 2\pi/\tau_\theta$ is the angular frequency and τ_θ the angular period. From (9) we can write for a small perturbation

$$\gamma = l'/l \approx 1 - \beta m/l^2 \qquad (23)$$

and

$$\langle\Omega\rangle \simeq \omega_\theta \beta m/l^2. \qquad (24)$$

By multiplying (19) and (20), and substituting (9), we have

$$l^2 \simeq mka^2b^2(1 + 2\beta/ka^2b^2). \qquad (25)$$

Substituting (25) into (24) we finally obtain

$$\langle\Omega\rangle \simeq \omega_\theta \frac{\beta}{ka^2b^2} + \mathrm{O}\left[\left(\frac{\beta}{ka^2b^2}\right)^2\right] \qquad (26)$$

which disagrees by a factor of 2 with Sivardière's result. This is due to an incorrect assumption for the angle between two successive periapses.

3. The oscillator under an arbitrary central perturbation

We will consider now the isotropic harmonic oscillator perturbed by an arbitrary central potential of the type β/r^n, where n is a non-zero integer. In this case the potential function is $U(r) = \frac{1}{2}kr^2 - \beta/r^n$ and the Binet formula reads

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{mk}{l^2u^3} + \frac{mn\beta u^{(n-1)}}{l^2} \qquad (27)$$

where $u = 1/r$.

Exact analytical solutions of (27) are not known except for a few cases and perturbative methods are unavoidable if we wish to obtain some knowledge about the mean precession rate $\langle\Omega\rangle_n$ and the polar equation of the orbit $u_n(\theta)$. Therefore let us consider the situation of stable approximately circular orbits. For a circular orbit the Binet equation yields

$$1 = mk/l^2u_0^4 + mn\beta u_0^{(n-2)}/l^2 \qquad (28)$$

where u_0 is the inverse of the radius of the circle r_0. In the case of small deviations from the circular orbit we can write

$$u(\theta) = u_0 + \delta(\theta) \qquad (29)$$

where $\delta(\theta)$ represents a small departure from the circular solution. Taking (29) into the Binet formula and using (28) we have

$$\mathrm{d}^2\delta/\mathrm{d}\theta^2 + \sigma_n\delta = \mathrm{O}(\delta^2/u_0^2) \qquad (30)$$

where σ_n is given by

$$\sigma_n = 4 - [mn(n+2)\beta u_0^{(n-2)}]/l^2. \qquad (31)$$

Note that, if $n(n+2)\beta$ is greater than 0, to guarantee the stability of the circular orbit the following condition must hold

$$mn(n+2)\beta u_0^{(n-2)}/l^2 < 4. \qquad (32)$$

On the other hand, if $n(n+2)\beta < 0$, stability is always assured. From now on we will consider only stable cases. Then for convenience we write $\sigma_n = C_n^2$. If we limit ourselves to a linear approximation in the ratio δ/u_0 we can write

$$\delta(\theta) = B\cos[C_n(\theta - \theta_0)] \qquad (33)$$

where B and θ_0 are constants of integration. For simplicity we take $\theta_0 = 0$; this represents our choice of the polar axis direction.

We can now calculate the mean rate of precession. If β is small we can rewrite C_n, to first-order approximation, as

$$C_n \simeq 2(1 - \varepsilon_n) \qquad (34)$$

where ε_n is given by

$$\varepsilon_n = [mn(n+2)\beta u_0^{(n-2)}]/8l^2. \qquad (35)$$

Therefore,

$$\delta_n(\theta) \simeq B\cos[2(1-\varepsilon_n)\theta]. \qquad (36)$$

For the perturbed oscillator ($\beta \neq 0$) the angle between two successive periapses is

$$\Delta\theta = 2\pi/2(1-\varepsilon_n). \qquad (37)$$

Note that $\varepsilon_n = 0$ (unperturbed oscillator) leads to $\Delta\theta = \pi$ as expected. Defining $\Delta\phi$ as the precession angle of the apoapsis (see figure 1), we have

$$\Delta\phi = \Delta\theta - \pi \simeq \pi\varepsilon_n. \qquad (38)$$

The mean rate of precession is

$$\langle\Omega\rangle_n \simeq \Delta\phi/\tau_{\theta/2} \simeq \omega_\theta\varepsilon. \qquad (39)$$

For $n = 2$ the perturbation is an inverse-cube force law and we obtain

$$\langle\Omega\rangle_2 = \omega_\theta m\beta/l^2 \qquad (40)$$

in agreement with (24).

Before concluding this section let us re-examine the polar equation of the orbit. Starting from (29) and keeping only terms up to first order in δ, we can write

$$r = r_0(1 + r_0\delta)^{-1} \simeq r_0 - \rho\cos(C_n\theta) \qquad (41)$$

where we have used (33) and defined $\rho = -r_0^2B$. The maximum and the minimum values of r, that we will identify with the major and the minor semi-axes of a precessing ellipse, are given respectively by $a = r_0 + \rho$ and $b = r_0 - \rho$. It follows that $B = (a-b)/2r_0^2$. Taking the square of a and b and summing we find that $a^2 + b^2 = 2r_0^2 + \mathrm{O}(\rho^2)$ and $a^2b^2 = r_0^4 + \mathrm{O}(\rho^2)$. Therefore, we can write

$$u_0^2 = 1/r_0^2 \simeq (a^2 + b^2)/2a^2b^2 \qquad (42)$$

and

$$2u_0B = (b^2 - a^2)/2a^2b^2. \qquad (43)$$

Now, taking the square of (29) and combining (33), (42) and (43), we obtain

$$u_n^2 \simeq \frac{a^2+b^2}{2a^2b^2} + \frac{b^2-a^2}{2a^2b^2}\cos(2\gamma_n\theta) \qquad (44)$$

where we have set $C_n = 2\gamma_n$. Using the trigonometric relation $\cos 2x = 2\cos^2 x - 1$, we finally obtain

$$u_n^2(\theta) \simeq \frac{\cos^2(\gamma_n\theta)}{a^2} + \frac{\sin^2(\gamma_n\theta)}{b^2}. \qquad (45)$$

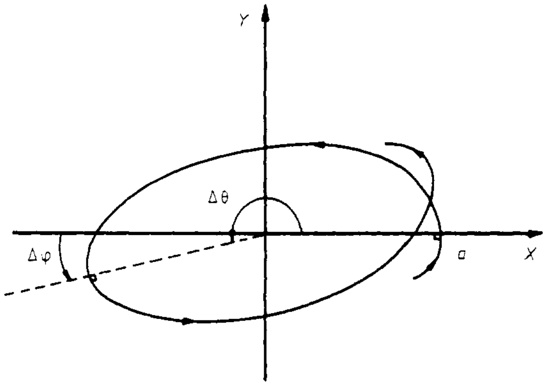

Figure 1. The angle of precession of the apoapsis $\Delta\varphi$ during a half-period of revolution.

What do all these algebraic manipulations mean? If n is equal to 2 we have proved that the orbit is a precessing ellipse. Note that this is true even when the orbit is far from circular. But if n is not equal to 2 and the strength of the perturbation weak enough so that all the approximations made are justified, any small departure from a stable circular orbit will also end in a precessing ellipse no matter the value of n. Of course, the mean rate of precession will be different—one for each value of n.

4. The charged oscillator in a constant magnetic field

We conclude by considering the case where the oscillator is placed in a constant magnetic field. It will be shown that, if the equations of motion can be treated in a perturbative way, it is possible to obtain the Larmor frequency without making use of inertial forces as is usual in many textbooks (see for example references [2–6]). For simplicity we choose $\boldsymbol{B}$ perpendicular to the plane of the orbit, $\boldsymbol{V}_0 \cdot \boldsymbol{B} = 0$, where V is the initial velocity. In polar coordinates the equations of motion are

$$m(\ddot{r} - r\dot{\theta}^2) = -kr - eBr\dot{\theta} \qquad (46)$$

and

$$(1/r)(\mathrm{d}/\mathrm{d}t)(mr^2\dot{\theta}) = eB\dot{r} \qquad \boldsymbol{B} = B(\hat{r}x\hat{\theta}). \qquad (47)$$

It follows from (47) that the angular speed $\dot{\theta}$ is given by

$$\dot{\theta} = \omega_L + c_1/mr^2 \qquad (48)$$

where c_1 is a constant of integration and $\omega_L = eB/2m$ is the Larmor frequency. If $B = 0$, then c_1 is the conserved angular momentum. If $\boldsymbol{B}$ is a weak field we can drop the term $eBr\dot{\theta}$ with respect to kr in (46).

Now we set $\Theta = \theta - \omega_L t$. With this change the equations of motion become

$$m\ddot{r} - mr\dot{\Theta}^2 = -kr \qquad (49)$$

$$\dot{\Theta} = c_1/mr^2. \qquad (50)$$

It is readily seen that the motion in the new variables is equivalent to the motion in the old variables without the magnetic field, in this case an ellipse with the geometrical centre at the centre of force. The new polar axis is defined by $\Theta = 0$, and hence $\theta = \omega_L t$, which means that the new polar axis rotates in the positive (counter-clockwise) sense with angular speed ω_L as shown in figure 1.

Therefore, when observed from the old reference frame the orbit will be a precessing ellipse with Larmor frequency ω_L.

5. Conclusions

In this work we have shown rigorously that the isotropic harmonic oscillator describes a precessing ellipse under an additional inverse-cube force law. For a weak inverse-cube perturbation we calculated the mean rate of precession. Next we treated the case of the isotropic oscillator under an arbitrary but still central perturbation of the form β/r^n. Using a perturbative scheme and starting from unperturbed orbits close to the circle, we were able to obtain the polar form of the perturbed orbit and the mean rate of precession. Finally, we discussed the case of the isotropic oscillator in a constant magnetic field. Using a perturbative approach again we obtained the Larmor frequency and showed that the perturbed trajectory is again a precessing ellipse. It is worth mentioning that the perturbative method employed here for the oscillator plus a constant magnetic field can be easily extended to the case of an arbitrary central force law plus a constant magnetic field.

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