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A note on the conservation of mechanical energy and the Galilean principle of relativity

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Abstract

A reexamination of simple examples that we usually teach to our students in introductory courses is the starting point for a discussion about the principle of conservation of energy and Galilean invariance.

1. Introduction

When students attend an intermediate course in classical mechanics most of them are introduced to gravity-assisted space flight or the gravitational catapult. Gravity-assisted space flight is a major advance in orbit design and basically it works in the following way. Consider a spacecraft of mass m that follows an orbital path and approaches a planet of mass M which is moving with a velocity \mathbf{V} with respect to some inertial reference system. Let \mathbf{v} and \mathbf{v}' be, respectively, the velocities of the spacecraft also measured with respect to this inertial frame of reference, initially at a point of the path far away from the planet and afterwards at a point also far away from it but now when the spacecraft is receding, as shown in figure 1. This is essentially a two-body problem with the planet acting as a moving centre of force. Moreover, we can assume that the motion of the planet is known beforehand, because $M \gg m$.

If we consider this two-body problem in an inertial reference system in which the planet is at rest and the initial and final velocities of the spacecraft are \mathbf{u} and \mathbf{u}' , respectively, then we will have $\|\mathbf{u}\| = \|\mathbf{u}'\| = u$, because it can be argued that with respect to the planet the energy of the spacecraft is conserved. Therefore, we write

$$\Delta E_{\text{spacecraft/planet}} = 0. \quad (1)$$

Let us now analyse the problem from the point of view of an inertial observer placed at some other location in the Solar System, for example, the Sun. Then, with respect to the Sun, the

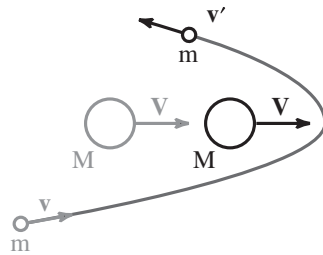


Figure 1. The gravitational catapult.

‘initial’ and ‘final’ velocities of the spacecraft are respectively $\mathbf{v} = \mathbf{u} + \mathbf{V}$, and $\mathbf{v}' = \mathbf{u}' + \mathbf{V}$. It follows that the energy variation of the spacecraft with respect to the Sun is

$$\Delta E_{\text{spacecraft/sun}} = \frac{1}{2} m v'^2 - \frac{1}{2} m v^2 = m(\mathbf{u}' - \mathbf{u}) \cdot \mathbf{V}. \quad (2)$$

Note that the spacecraft can gain or lose energy in the process. It all depends how it approaches the planet. This apparent breaking of the energy conservation law due to a simple change of the reference system is certain to turn on a student’s attention. It is usually explained away by stating that the planet must also lose or gain energy [1], although its speed remains constant! But, is this statement compatible with the $M \gg m$ hypothesis? Is it compatible with Galilean invariance? Why is this gain or loss of energy and momentum not taken into account in the previous analysis when the planet was at rest? Later on this same type of questioning reappeared when we were dealing with similar problems. One of our aims here is to discuss these questions.

The law of conservation of energy is one of the most fundamental laws of physics. In the context of Newtonian mechanics, for an isolated mechanical system, this law assumes a particular form and we call it the law of conservation of mechanical energy and by this we mean that the sum of the kinetic energy and the potential energy is constant in time. On the other hand, the laws of Newtonian physics obey the Galilean principle of relativity, that is, they are invariant under the transformations of the Galilean group of transformations which encompass rotations, translations and boosts. It follows that all theorems stemming from Newton’s laws, including the principle of conservation of energy, must also be compatible with the Galilean principle of relativity, that is, they must hold in all inertial reference systems.

The same is true for the conservation of linear momentum or the conservation of the angular momentum. Nevertheless, it is not difficult to find in modern texts on basic physics examples where the initially assumed law of conservation of mechanical energy seems not to hold in all inertial systems, which is not conceptually acceptable. A more careful examination of those examples shows that what is being overlooked is the issue of the compatibility of the application of the conservation law to the example at hand and Galilean invariance. In modern basic physics textbooks, for instance [2, 3], it is usually correctly stressed that in order to have conservation of energy, the system must be isolate, but we do not see the conservation of mechanical energy and its invariance under Galilean transformations discussed simultaneously. This together with assumptions not clearly stated can lead us to puzzling situations in which the conservation law for the energy seems to hold only in some preferred reference systems. The following two simple examples illustrate our point.

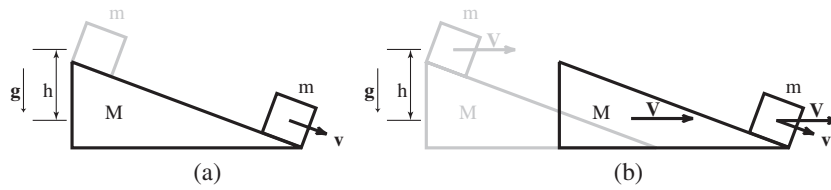


Figure 2. Wedge and block seen from two different inertial frames. The small block of mass m slides without friction on the surface of the wedge that stays put. This is equivalent to saying that $M \rightarrow \infty$ or $M \gg m$. Under what conditions can we consider the wedge as a valid inertial frame?

2. Two simple examples: wedge and block and free fall

Consider a small block whose mass is m sliding on the inclined surface of a fixed wedge (with respect to the Earth) whose mass is M , as in figure 2. By ‘fixed’ we mean that $M \rightarrow \infty$, or $M \gg m$. Suppose we want to know the speed of the block when it leaves the wedge. Let us analyse the problem from the point of view of a reference system fixed with respect to the wedge. Let h be the initial height of the small block and for the sake of simplicity let us also suppose that its initial velocity with respect to the wedge is zero. The usual solution we teach to our students is based on the principle of conservation of the mechanical energy applied to the block considered as a system. The reasons to invoke this principle are as follows:

- (a) there is no friction between the surfaces in contact of the wedge and the block; and
- (b) the normal force on the block does not perform work.

Then, it follows that

$$mgh = \frac{1}{2}m\mathbf{v}^2, \quad (3)$$

where \mathbf{v} is the velocity of the block at the base of the wedge. Solving this equation for $v = \|\mathbf{v}\|$, we obtain

$$v = \sqrt{2gh}. \quad (4)$$

Let us consider now the same problem analysed from the point of view of an inertial reference system moving with constant velocity $-\mathbf{V}$ with respect to the wedge and parallel to its base. The initial velocity of the block in this second reference system is \mathbf{V} and its final velocity is $\mathbf{v} + \mathbf{V}$. Therefore, its initial energy will be

$$E_0 = \frac{1}{2}m\mathbf{V}^2 + mgh, \quad (5)$$

and its final energy

$$E_1 = \frac{1}{2}m(\mathbf{v} + \mathbf{V})^2. \quad (6)$$

It easily follows that the variation of the mechanical energy of the small block is

$$\Delta E = m\mathbf{v} \cdot \mathbf{V}. \quad (7)$$

This last equation shows clearly that the energy of the block is not conserved in the second inertial system. Note also that we can no longer state that the contact forces do not perform work. Moreover, it should be clear that the block is not an isolated mechanical system because it is subjected to external forces: the contact force that the wedge exerts on it and its weight. We can easily calculate the work done by the contact force and compare the result with

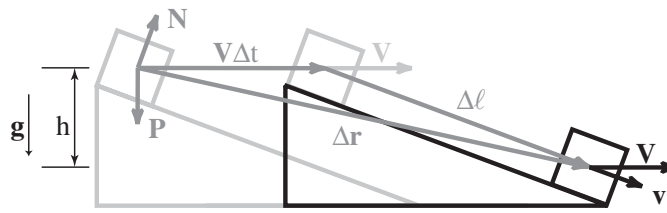


Figure 3. The vector $\Delta \mathbf{r}$ is the displacement of the centre of mass of the block as seen from the point of view of an inertial reference system moving with constant velocity $-\mathbf{V}$ with respect to the wedge. We also indicate the external forces that act on the block.

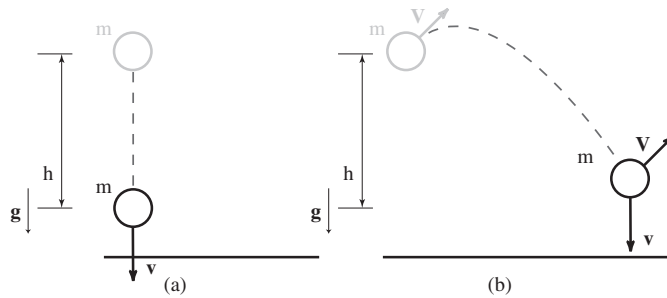


Figure 4. Free fall seen from two reference systems. Here $-\mathbf{V}$ is the velocity of the second inertial reference system with respect to the first.

equation (7). The work done by the normal force \mathbf{N} is

$$W_N = \mathbf{N} \cdot \Delta \mathbf{r}, \quad (8)$$

where $\Delta \mathbf{r}$ is the displacement of the centre of mass of the block as seen from the point of view of an inertial reference system moving with constant velocity $-\mathbf{V}$ with respect to the wedge. The displacement can be written as

$$\Delta \mathbf{r} = \mathbf{V} \Delta t + \Delta \ell, \quad (9)$$

where $\Delta \ell$ is the displacement of the block along the surface of the wedge, see figure 3. Because \mathbf{N} is perpendicular to $\Delta \ell$, the work of the normal force is

$$W_N = \mathbf{N} \cdot \mathbf{V} \Delta t. \quad (10)$$

From the equation of motion of the block, we have

$$\mathbf{N} = m\mathbf{a} - \mathbf{P}; \quad (11)$$

hence

$$W_N = (m\mathbf{a} - \mathbf{P}) \cdot \mathbf{V} \Delta t = m\mathbf{v} \cdot \mathbf{V}, \quad (12)$$

where we have made use of $\mathbf{a} \Delta t = \mathbf{v}$, and of the fact that $\mathbf{P} = m\mathbf{g}$ is perpendicular to the velocity of the wedge. This result is in agreement with the one given by equation (7).

As a second simple example, consider the familiar free fall problem, see figure 4. Let a particle of mass m be released vertically from a height h above the surface of the Earth and let us calculate its speed immediately before it hits the ground. Most of us argue that since the mass of the Earth is enormous when compared to the mass of the free falling particle, it

is a good approximation to consider the Earth as an inertial reference system. From this the solution follows immediately. We write

$$mgh = \frac{1}{2}m\mathbf{v}^2, \quad (13)$$

and from this we obtain $v = \|\mathbf{v}\|$.

Let us consider the same problem from the point of view of an inertial reference system whose velocity with respect to the Earth is $-\mathbf{V}$. The initial mechanical energy of the particle is

$$E_0 = \frac{1}{2}m\mathbf{V}^2 + mgh, \quad (14)$$

and its final energy is

$$E_1 = \frac{1}{2}m(\mathbf{v} + \mathbf{V})^2. \quad (15)$$

The mechanical energy variation now reads

$$\Delta E = m\mathbf{v} \cdot \mathbf{V}. \quad (16)$$

Once again the conservation of mechanical energy holds in a special reference system, the one in which the Earth stands still.

3. The limit $M \rightarrow \infty$

In order to shed light on the matter, let us consider the problem from a general point of view. Let there be a system of $N + M$ particles each with a mass m_i , $i = 1, \dots, N + M$. Suppose that the system is isolated and also that the internal forces can be classified into two sets, namely the set of conservative forces and the set of forces whose total work is zero. Denoting by \mathbf{x}_i the position and by \mathbf{v}_i the velocity of particle of mass m_i with respect to an arbitrarily chosen inertial reference system, we write the total mechanical energy E , the total linear momentum \mathbf{P} and the total angular momentum of the mechanical system \mathbf{L} , respectively, as

$$E = \sum_{i=1}^{N+M} \frac{1}{2}m_i\mathbf{v}_i^2 + U(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N+M}), \quad (17)$$

where $U(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N+M})$ is the total internal potential energy of the system

$$\mathbf{P} = \sum_{i=1}^{N+M} m_i\mathbf{v}_i \quad (18)$$

and

$$\mathbf{L} = \sum_{i=1}^{N+M} \mathbf{x}_i \times m_i\mathbf{v}_i. \quad (19)$$

Let us divide the system into two subsystems, see figure 5(a): subsystem A formed by the $i = 1, \dots, N$ particles and subsystem B formed by the $N + 1, \dots, N + M$ remaining particles. In this way the total mechanical energy given by equation (17), the total linear momentum given by equation (18) and the angular momentum given by equation (19) can be decomposed in the following way:

$$E = T_A + T_B + U_A + U_B + U_{AB}, \quad (20)$$

$$\mathbf{P} = \mathbf{P}_A + \mathbf{P}_B, \quad (21)$$

$$\mathbf{L} = \mathbf{L}_A + \mathbf{L}_B, \quad (22)$$

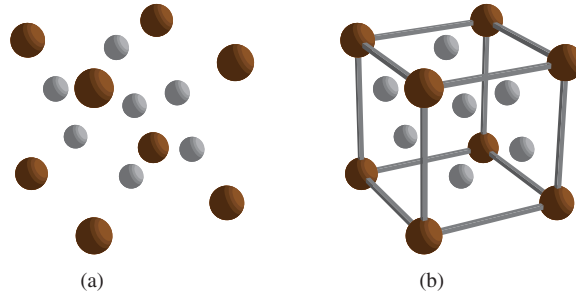


Figure 5. System formed by $i = 1, \dots, N$ particles (subsystem A , in light grey colour) plus $N + 1, \dots, N + M$ remaining particles (subsystem B , in dark rusty colour).

(This figure is in colour only in the electronic version)

where $T_{A(B)}$ is kinetic energy of system A (B), $U_{A(B)}$ is the potential energy of A (B) and U_{AB} is the interaction potential between the two subsystems. Recall that the complete system is isolated; therefore, these mechanical quantities are conserved, that is

$$\frac{d}{dt} (T_A + U_A + U_{AB}) + \frac{d}{dt} (T_B + U_B) = 0, \quad (23)$$

$$\frac{d\mathbf{P}_A}{dt} + \frac{d\mathbf{P}_B}{dt} = 0, \quad (24)$$

$$\frac{d\mathbf{L}_A}{dt} + \frac{d\mathbf{L}_B}{dt} = 0. \quad (25)$$

Equation (23) can be rewritten as

$$\frac{d}{dt} (T_A + U_A + U_{AB}) = -\frac{d}{dt} \left(\frac{1}{2} M_B \mathbf{V}_B^2 + T'_B \right) - \frac{dU_B}{dt}, \quad (26)$$

where M_B and \mathbf{V}_B are the mass and centre of mass velocity of subsystem B , respectively, with respect to the inertial reference system, and T'_B is its kinetic energy with respect to the centre of mass. In order to consider subsystem B as a reference system suppose that the particles belonging to B are rigidly linked one to the other, see figure 5(b). This means that its internal potential energy is constant, $U_B = \text{constant}$, and the kinetic energy is given by

$$T'_B = \frac{1}{2} \boldsymbol{\omega}_B \cdot \mathbf{I} \cdot \boldsymbol{\omega}_B, \quad (27)$$

where we have introduced the angular velocity vector $\boldsymbol{\omega}_B$ and the inertia tensor \mathbf{I} of the subsystem B . Expanding the rhs of equation (26) we obtain

$$\frac{d}{dt} (T_A + U_A + U_{AB}) = -\frac{d\mathbf{P}_B}{dt} \cdot \mathbf{V}_B - \frac{d\mathbf{L}'_B}{dt} \cdot \boldsymbol{\omega}_B, \quad (28)$$

where \mathbf{L}'_B is the angular momentum of subsystem B with respect to its own centre of mass. We can write \mathbf{L}'_B in the form

$$\mathbf{L}'_B = \mathbf{L}_B - \mathbf{R}_B \times \mathbf{P}_B, \quad (29)$$

where \mathbf{R}_B is the position of the centre of mass of subsystem B with respect to the arbitrary inertial reference system that we have chosen in the beginning. The time derivative of \mathbf{L}'_B is given by

$$\frac{d\mathbf{L}'_B}{dt} = \frac{d\mathbf{L}_B}{dt} - \mathbf{R}_B \times \frac{d\mathbf{P}_B}{dt}. \quad (30)$$

Taking this result into equation (28), we get

$$\frac{dE_{AB}}{dt} = -\frac{d\mathbf{P}_B}{dt} \cdot \mathbf{V}_B - \left(\frac{d\mathbf{L}_B}{dt} - \mathbf{R}_B \times \frac{d\mathbf{P}_B}{dt} \right) \cdot \boldsymbol{\omega}_B, \quad (31)$$

where we have defined the mechanical energy of subsystem A with respect to subsystem B by

$$E_{AB} := T_A + U_A + U_{AB}. \quad (32)$$

Note that this definition corresponds exactly to the one we made use of in the examples. For the block plus fixed wedge system, for instance, $U_A = \text{constant}$, T_A is the kinetic energy of the block and $U_{AB} = mgh$ is the interaction energy between the moving block and the fixed wedge plus Earth system that we can identify as subsystem B . Making use of equations (24) and (25) and performing additional manipulations, we obtain

$$\frac{dE_{AB}}{dt} = \mathbf{F}_A \cdot \mathbf{V}_B + (\boldsymbol{\tau}_A - \mathbf{R}_B \times \mathbf{F}_A) \cdot \boldsymbol{\omega}_B, \quad (33)$$

where $\mathbf{F}_A = d\mathbf{P}_A/dt$ and $\boldsymbol{\tau}_A = d\mathbf{L}_A/dt$ are, respectively, the total force and total torque exerted on subsystem A by subsystem B . In equation (33), the only quantities related to subsystem B are kinematical. Moreover, \mathbf{R}_B , \mathbf{V}_B and $\boldsymbol{\omega}_B$ depend in principle on time. In order to consider subsystem B as an inertial reference system \mathbf{V}_B must be equal to a constant and $\boldsymbol{\omega}_B$ must be zero. In this case, taking into account equations (21) and (22), we see that these conditions can be implemented only if the mass of subsystem B is infinite. Even if subsystem B behaves as an inertial reference system in this limit, equation (33) reduces to

$$\frac{dE_{AB}}{dt} = \mathbf{F}_A \cdot \mathbf{V}_B, \quad (34)$$

and this means that the energy E_{AB} is not constant when measured with respect to an arbitrary inertial system that moves with a constant velocity, though the total energy is. We still have to impose an additional condition, namely \mathbf{V}_B must be zero or orthogonal to \mathbf{F}_A , and this will be true only for a special inertial system. Only in this particular inertial system where both conditions are satisfied, we will have

$$E_{AB} = \text{constant}. \quad (35)$$

The examples we dealt with in the beginning and many others follow the same line of reasoning described here and this is the reason why they appear to violate the law of conservation of energy when we perform a perfectly admissible Galilean boost. Equation (35) cannot be considered as a conservation law for the mechanical energy because it holds only in the special inertial reference system for which the velocity of subsystem B is zero. One must always keep in mind that a true conservation law must hold in all inertial reference systems. Moreover, equation (32) that defines this energy contains the term U_{AB} that describes the interaction energy of the two systems; hence, it cannot be interpreted as the total energy of subsystem A .

4. Final remarks

In this paper we have discussed the application of the principle of conservation of energy to mechanical systems. Through simple examples, we have shown that paradoxical situations may arise if proper care concerning its application is not taken. There are more similar examples that we could have considered here, such as the massless spring or the collision between a particle and a moving wall. They all lead to the same conclusion: if we apply a Galilean transform to them, energy is not conserved in the new inertial reference system. Nevertheless, it must be strongly emphasized that the validity of the conservation of mechanical

energy is not being questioned here; in contrast, our firm belief in it is the reason why we have refrained from considering empirical arguments. What we have argued for is that the application of the concept of conservation of mechanical energy requires a careful procedure. In particular, we have stressed the fact that the conservation of mechanical energy, as well as conservation of linear momentum and angular momentum, should not depend on our choice of any particular inertial reference system.

The mechanical systems we have analysed here are frequently found in textbooks as examples of conservation of energy, in the way they are presented; however, they are not isolated systems, see equation (33). As a consequence, physical quantities associated with them, such as linear momentum, angular momentum or mechanical energy, are not in general constant, but the energy may be constant in some special inertial frame of reference. We can summarize our results by answering the following question: How should we approach the teaching of this important topic of the physics syllabus in order to avoid conceptual problems? If we want to be on the safe side, one possibility is to make use of the work–kinetic energy theorem that follows from the integration of the Newtonian equation of motion in a particular reference system. The other possibility is the one we have advocated for here, that is: consider the system as a whole and no conceptual problems will come in our way. There is no need to be formal. The formal procedure we have developed here can be easily adapted and applied to the examples in a straightforward way.

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