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has the same expressions for $E$ and $B$ as in Eqs. (1) and (2).

[10]Reference 6, p. 206, prob. 5.7 and p. 697, prob. 14.12 and 14.13. There are no radiation and relativistic corrections to the static $E$ and $B$ of a steady, *uniform* current in a closed path.

[11]This can be easily proved by integrating the relation $dm = i(\theta)dA$ over the spherical surface with uniform surface charge density $\sigma = Q/4\pi a^2$ to obtain $m$ and finally obtain $M = m/(\frac{4}{3}\pi a^3)$.

[12]Having identified the electromagnetic MI as in Eq. (9), it can now be easily shown that Eq. (10) holds for the charged magnetized sphere of Sec. III as well.

[13]For a similar but quantum picture see Hans C. Ohanian, Am. J. Phys. **54**, 500 (1986) and J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[14]See Reference 1, Sec. 28-2. The result for the electromagnetic mass of the electron derived there does not alter upon inclusion of the magnetic field due to the magnetic moment of the electron. Note that the magnetic moment of the electron is assumed to be uniformly spread out over a sphere of radius $a$.

# Elastic scattering by a paraboloid of revolution

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The differential cross section for scattering by a perfectly elastic, impenetrable paraboloid of revolution is obtained. The angular dependence is identical to that for Rutherford scattering. It follows that Rutherford scattering of particles of a particular energy is equivalent to scattering from a particular paraboloid of revolution.

## I. THE SCATTERING CROSS SECTION

Let a stream of particles be incident upon a hard (i.e., elastic and impenetrable) paraboloid of revolution, with velocity vectors initially parallel to the axis of the paraboloid. Figure 1 represents a plane section containing the axis of the paraboloid. The figure shows the trajectory of a typical particle, incident with impact parameter $b$. Because of the familiar focusing property of the parabola, the particle will be reflected in such a way that it will appear to have come from the focus. The final direction of the scattered particle makes an angle $\theta$ with the initial line of motion.

The equation of the parabolic section, in a system of plane polar coordinates with origin at the focus of the parabola, is $r = \alpha/(1-\cos\theta)$, where $\alpha$ is half the latus rectum, as illustrated, and $\theta$ is the scattering angle. Or, since the focal length $f$ is equal to $\alpha/2$,

$$r = 2f/(1-\cos\theta).$$

With the equation in this form, $r$ is a minimum at $\theta = \pi$, as the figure requires.

The differential scattering cross section is

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|.$$

Thus we must obtain an expression for $b$ in terms of $\theta$. From the figure,

$$b = r\sin\theta$$
$$= 2f\sin\theta/(1-\cos\theta).$$

Thus

$$\frac{db}{d\theta} = \frac{-2f}{(1-\cos\theta)}.$$

Substitution of these expressions for $b$ and $db/d\theta$ in the formula for the cross section (with use of a trigonometric identity) gives

$$\sigma(\theta) = f^2 \sin^{-4}(\theta/2) \quad \text{(Hard paraboloid)}. \tag{1}$$

## II. COMPARISON WITH THE RUTHERFORD PROBLEM AND WITH THE HARD SPHERE

The angular distribution of the scattered particles is exactly the same as in the case of electrostatic scattering of charged particles by a fixed point charge. For this electro-

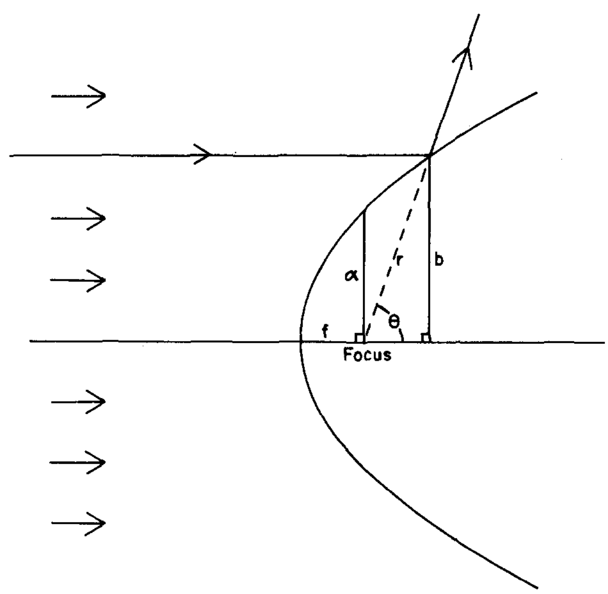


Fig. 1. Scattering from an elastic, impenetrable paraboloid of revolution.

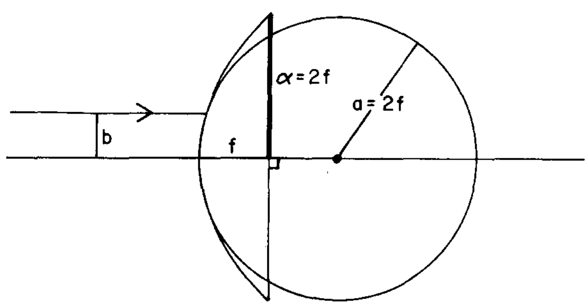


Fig. 2. A truncated paraboloid of revolution with semilatus rectum $\alpha$, superimposed on a sphere with radius $a = \alpha$. The paraboloid has been truncated by a plane through its focus perpendicular to the axis.

static case, the Rutherford scattering formula is

$$\sigma(\theta) = (k^2/16T_0^2) \sin^{-4} (\theta/2) \text{ (Rutherford formula)}, \quad (2)$$

where $T_0$ is the kinetic energy of the incident particles when they are far from the force center and $k$ is the constant in the Coulomb force law ($F = k/r^2$). The scattering cross section (1) for the paraboloid of revolution has, of course, no energy dependence, as the scattering process involves only contact forces at a hard surface.

It is also fruitful to compare Eq. (1) with the differential cross section for scattering by a hard sphere. The sphere scatters isotropically, so $\sigma$ is independent of $\theta$:

$$\sigma(\theta) = a^2/4 \quad \text{(Hard sphere)}, \quad (3)$$

where $a$ is the radius of the sphere. Let a paraboloid of revolution be truncated by a plane through the focus and perpendicular to the axis, as shown in Fig. 2. The semilatus rectum $\alpha$ ( $= 2f$ ) of the parabola is then the effective radius of the finite target. A sphere of radius $a = 2f$ has the same effective size as the truncated paraboloid. Moreover, the radius of curvature of the paraboloid at small impact parameter $b$ is indistinguishable from that of the sphere. The total scattering cross sections of the two targets will be exactly the same, $\pi a^2$. Moreover, the differential scattering cross sections will be nearly the same for large scattering angles (see Fig. 3). At small scattering angles, the two targets may easily be distinguished: $\sigma(\theta)$ is constant for the sphere; for the truncated paraboloid, $\sigma(\theta)$ follows the $\sin^{-4}(\theta/2)$ law between $\theta = \pi$ and $\pi/2$, but the cross section is zero for scattering angles less than $\pi/2$.

The author has found this simple problem a useful addition to a more or less standard treatment of scattering theory in a mechanics course at the upper undergraduate level. Rutherford scattering and scattering from a hard sphere are almost invariably treated in such courses.[1-4] The hard paraboloid stands between these two classic examples: The differential cross section for scattering from the hard paraboloid shares the angular dependence of the Rutherford problem and the energy independence of the hard sphere problem. The three cross sections taken together thus give the opportunity for some useful thought experiments: What kind of measurements would the student suggest for determining which of the three possible targets was hidden in a black box? Exercises of this kind have proved helpful in developing a better understanding of cross sections by beginning students.

## III. COROLLARY ON RUTHERFORD SCATTERING

The similarity of Eqs. (1) and (2) implies that the electrostatic scattering of charged particles of a particular energy is equivalent to contact scattering from a particular paraboloid of revolution.

In Fig. 4, a fixed electric charge is placed at 0. Let a stream of particles with charges of the same sign be incident from the left with velocity vectors initially parallel to the line AA′ through 0. The hyperbolic trajectory of one such particle is shown. The line of initial motion is BB′, which is one asymptote of the hyperbola. The other asymptote is DD′. The scattering angle $\theta$ is the angle between the two asymptotes.

Draw CC′ through 0 parallel to DD′. At P, CC′ intersects BB′. Now, compare Fig. 4 (for Rutherford scattering) with Fig. 1 (for the hard paraboloid of revolution). In Fig. 4 BP corresponds to the track of the particle before reflection, and PC′ to its track after reflection from the paraboloid. Thus, in the case of Rutherford scattering, it is clear that the locus of all such points P (defined as in Fig.

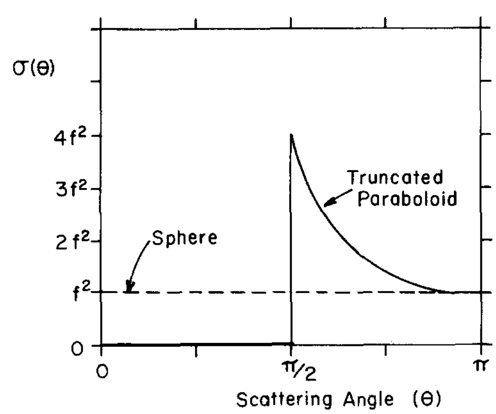


Fig. 3. Differential scattering cross sections for the two targets illustrated in Fig. 2.

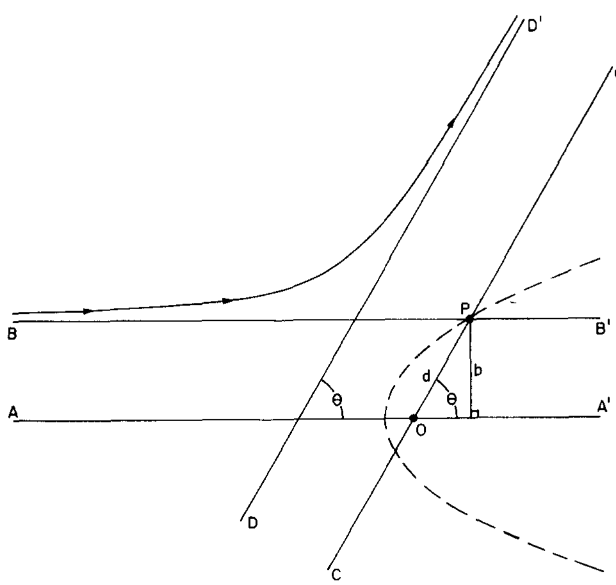


Fig. 4. A Rutherford scattering situation and the equivalent paraboloid.

4) is a paraboloid of revolution. The equivalent parabola has been sketched in a broken line. The paraboloid is equivalent to the fixed charge in the following sense. If the charge at 0 were suddenly removed and the paraboloid inserted in its place, the angular distribution of the scattered particles would not be changed.

The equation of the equivalent paraboloid is easily derived. In Fig. 4, denote OP, the distance from the scattering center to the effective paraboloid, by $d$. Then

$$d = b/\sin\theta.$$

The connection between the impact parameter $b$ and the scattering angle $\theta$ for the Rutherford problem is[5]:

$$b = k/2T_0 \tan(\theta/2),$$

where $k$ and $T_0$ have the same meaning as before. Elimination of $b$ between these equations and use of trigonometric identities gives

$$d = k/2T_0(1-\cos\theta),$$

which is the equation of the effective parabola. Rutherford scattering of particles of energy $T_0$ is, therefore, equivalent to the scattering of elastic particles by a hard paraboloid of revolution of semilatus rectum $k/2T_0$. This result is apparent also by comparison of formulas (1) and (2). The effective paraboloid is illustrated in Fig. 4: It is the locus of all points P, each of which is the intersection of the asymptote of an incident particle with the line through the focus parallel to the asymptote of the same particle on the way out.


[1]J. B. Marion, *Classical Dynamics of Particles and Systems* (Academic, New York, 1970), 2nd ed., pp. 302–311.

[2]K. R. Symon, *Mechanics* (Addison-Wesley, Reading, MA, 1971), 3rd ed., pp. 137–140, 157.

[3]L. D. Landau and E. M. Lifshitz, *Mechanics*, English trans. by J. B. Sykes and J. S. Bell (Pergamon, Oxford, 1976), 3rd ed., pp. 48–57.

[4]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980), 2nd ed., pp. 105–114.

[5]See, for example, Ref. 1, p. 308; or Ref. 2, p. 138.


# Gauge invariance and quantization


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Quantizing theories such as quantum electrodynamics that contain a gauge invariance are discussed via a simple pedagogical example. Canonical and path integral quantization methods are compared.


As is well known, the idea of requiring invariance under *local* group transformations has proved to be an extraordinarily productive one in physics. Thus demanding local $U(1)$ invariance of the Dirac Lagrangian produces the theory of quantum electrodynamics[1] (QED) while local $SU(2)_L \times U(1)$ symmetry leads to the Weinberg–Salam model of the electroweak interaction.[2] Also, local SU(3) gives rise to "quantum chromodynamics," which may well describe the strong interactions.[3] In each of these cases one deals with a so-called gauge symmetry, in which one has classes of models that are totally equivalent to one another, being related by a simple redefinition of the fields called a gauge transformation. This feature, however, implies that subtleties arise in the quantization of such theories, which often are the source of confusion to students. The usual model in which gauge symmetry is introduced—quantum electrodynamics—involves all the complications of quantum field theory and for this reason it is pedagogically efficacious to introduce these concepts within a simpler framework, which is the topic of this article.

Thus we consider a model Lagrangian of the form[4]

$$L = (m/2)(\dot{x}_1^2 + \dot{x}_2^2) - m\xi(x_1\dot{x}_2 - x_2\dot{x}_1) + \tfrac{1}{2}m\xi^2(x_1^2 + x_2^2) - V(x_1^2 + x_2^2), \quad (1)$$

where $x_1, x_2$ are Cartesian coordinates. Here, $\xi$ is another coordinate, but $\dot{\xi}$ does not appear in the Lagrangian. Equivalently, we can write this Lagrangian in polar coordinate form,

$$L = (m/2)[\dot{r}^2 + r^2(\dot{\theta} - \xi)^2] - V(r), \quad (2)$$

where we observe that $L$ is invariant under the redefinitions

$$\theta \to \theta' \equiv \theta + \chi(t), \quad \xi \to \xi' \equiv \xi + \dot{\chi}(t), \quad (3)$$

where $\chi$ is an arbitrary function of time. Models using the coordinates $\theta$, $\xi$ must contain the same physics as those employing $\theta'$, $\xi'$. This, then, is a simple example of a gauge invariance. Our example is, of course, somewhat artificial. Its physical significance may be gleaned by introducing a third orthogonal direction $\hat{x}_3$ and defining

$$\boldsymbol{\xi} = \xi\hat{x}_3. \quad (4)$$

Then, with

$$\mathbf{r} = x_1\hat{x}_1 + x_2\hat{x}_2, \quad (5)$$

the Lagrangian can be written as

$$L = (m/2)(\dot{\mathbf{r}} - \boldsymbol{\xi}\times\mathbf{r})^2 - V(r), \quad (6)$$

which describes a particle of mass $m$ confined to a plane and moving under the influence of a potential $V(r)$ as seen by an observer in a (noninertial) frame rotating with angu-