Kepler's Laws: Demonstration and Derivation without Calculus

Seville Chapman

Citation: American Journal of Physics **37**, 1134 (1969); doi: 10.1119/1.1975231
View online: https://doi.org/10.1119/1.1975231
View Table of Contents: https://aapt.scitation.org/toc/ajp/37/11
Published by the American Association of Physics Teachers

ARTICLES YOU MAY BE INTERESTED IN

Elementary derivation of Kepler's laws
American Journal of Physics **64**, 392 (1996); https://doi.org/10.1119/1.18253

An unusual approach to Kepler's first law
American Journal of Physics **69**, 1036 (2001); https://doi.org/10.1119/1.1379735

The conservation principles and Kepler's laws of planetary motion
American Journal of Physics **43**, 575 (1975); https://doi.org/10.1119/1.9770

Kepler's laws and universal gravitation in Newton's Principia
American Journal of Physics **50**, 617 (1982); https://doi.org/10.1119/1.13059

Kepler's third law
American Journal of Physics **49**, 691 (1981); https://doi.org/10.1119/1.12430

Kepler's Third Law for Elliptical Orbits
American Journal of Physics **30**, 813 (1962); https://doi.org/10.1119/1.1941813

AAPT 2021
PHYSICS EDUCATION®
VIRTUAL WINTER MEETING
JANUARY 9–12

where $f=1+O(v)$. The equivalence of different inertial systems requires that the reverse transformations have the same form, with primed and unprimed quantities exchanged. The arbitrary factors are F', V', and f'. Then we find

$$F'/f=f'/F=-v/v',$$

$$v^2/V^2=v'^2/V'^2,$$

$$FF'=ff'=[1-(v^2/V^2)]^{-1}. \quad (10)$$

Up to this point, we have used the equivalence of systems only in the restricted sense that we can transform between arbitrary systems in a similar way. Now we must use this principle in a stronger form. We assume that the transformation shall not prefer any one system to another. This implies firstly that $v'=-v$, and secondly, that the transformation does not depend on the direction, i.e., $F=F(v^2)$ etc. Note that it is this assumption which ultimately rules out the possibility of zero-order constants being system-dependent. Therefore, any "absolute" motion of systems cannot be observed. We now have

$$F=F'=f=f'=(1-v^2/V^2)^{-1/2}. \quad (11)$$

The result is a Lorentz transformation with the invariant velocity V which is equal to the zero-order value of $(\epsilon_0\mu_0)^{-1/2}$. It follows that this quantity, or the velocity c of light *in vacuo,* is itself invariant since it equals the invariant velocity in one system, namely, the rest system.

ACKNOWLEDGMENTS

I am indebted to the European Space Research Organization for a NASA International University Fellowship in Space Science.

AMERICAN JOURNAL OF PHYSICS VOLUME 37, NUMBER 11 NOVEMBER 1969

Kepler's Laws: Demonstration and Derivation without Calculus

SEVILLE CHAPMAN

Cornell Aeronautical Laboratory, Buffalo, New York 14221

(Received 24 March 1969)

A demonstration apparatus for Kepler's three laws of planetary motion consists of an air-supported "satellite" whose orbit on a 1.2×1.5-m level-table surface is determined by an inverse-square force generated by a Peaucellier linkage and long spring. The equipment is a feasibility model which works—not an ultimate design. Cost of the equipment is about $25 for parts, plus considerable labor. Several comments are made regarding tricky aspects of the design. The apparatus can be analyzed by using high-school algebra and geometry. From the principles of conservation of energy, conservation of momentum, and static equilibrium, Kepler's laws can be deduced that: (1) the oribt is an ellipse with center of force at one focus, (2) the time rate of area swept out by the radius vector is constant, and (3) the square of the period is proportional to the cube of the semimajor axis. This analysis (believed to be new) in essence integrates the force, $1/r^2$, to obtain the energy, $-1/r$, without using calculus.

INTRODUCTION

Motion of a body under an inverse-square, central-force law has application in physics from atomic to galactic phenomena. In 1618 Kepler stated his three Laws of Planetary Motion from which Newton, about 1680, using the notion of limits, deduced the inverse-square law of gravitation. Now in the space age there is new interest in ellipses and Kepler's laws[1–12] as applied to satellites and ballistic missiles. The special case of circular orbits is sometimes considered in elementary courses.

In this paper we show that the mathematical expressions for elliptic orbits can be derived with no more than high-school algebra and geometry.

The part believed to be new is a method involving a famous simple linkage, which shows that an inverse-square force law results in an inverse-first-power energy relationship without using calculus.

MAIN ARGUMENT

The main mathematical and physical argument is very brief. For a body of mass m subject to an inverse-square force law,

$$F=GMm/r^2, \tag{1}$$

where F is the force of attraction to the center of force, r is the radial distance to the body, and GM is a constant. Its potential energy is shown (see Appendix A) to be

$$U=-GMm/r. \tag{2}$$

Since its kinetic energy is $0.5\,mv^2$ where v is its speed, its total energy (which is negative to a

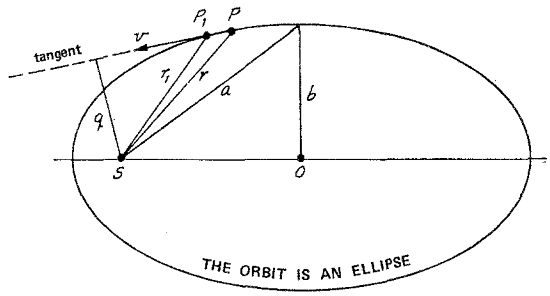


FIG. 1. Elliptic orbit of a satellite at point P, around a focus S.

zero value at infinity) is

$$0.5mv^2-GMm/r=\text{const}=-K_1, \tag{3}$$

providing the principle of conservation of energy applies (i.e., no friction). For a central force there are no torques, so that angular momentum (of a satellite presently at a point P, see Fig. 1) is another constant. Thus

$$mqv=\text{const.}=K_2, \tag{4}$$

where q is the perpendicular distance from the center of force to the line of the velocity. Whence, substituting for v from Eq. (4) in Eq. (3),

$$0.5K_2^2/K_1mq^2-GMm/K_1r=-1, \tag{5}$$

which is of the form

$$b^2/q^2-2a/r=-1, \tag{6}$$

which is the well-known (i.e., not absolutely unknown) pedal equation of the ellipse having semimajor axis a, semiminor axis b, with radial distance from the focus r, and pedal (or perpendicular) distance q. The center of force is clearly at one of the foci. Certain aspects of this derivation have been used by several authors.[13–19]

[1] Franklin Miller, Jr., Amer. J. Phys. **34**, 53 (1966).

[2] James L. Cronin, Jr. and L. C. Jones, Amer. J. Phys. **35**, 219 (1967).

[3] James L. Cronin, Jr. and L. C. Jones, Amer. J. Phys. **36**, 1016 (1968).

[4] John Harris and Fletcher Watson, Phys. Teach. **6**, 394 (1968).

[5] Seville Chapman, *The Ellipse* (CAL Report No. 168, April 1967).

[6] Roy C. Spender, Amer. Rocket Society J. **31**, 158 (1961).

[7] D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination* (Chelsea Publ. Co., New York, 1952). Describes the pedal construction of an ellipse.

[8] Martin Gardner, *New Mathematical Diversions* (Simon and Schuster, Inc., New York, 1966), Chap. 15 on the ellipse.

[9] Donavan A. Johnson, *Paper Folding for the Mathematics Class* (National Council of Teachers of Mathematics, Washington, D.C., 1957).

[10] Robert C. Yates, *Geometrical Tools* (Educational Publishers, Chicago, 1963).

[11] Robert C. Yates, *Curves* (J. W. Edwards, Ann Arbor, Mich., 1947).

[12] George Salmon, *A Treatise on Conic Sections* (Chelsea Publ. Co., New York, 1954), 6th ed., especially pp. 160–171 on the ellipse.

[13] E. H. Lockwood, *A Book of Curves* (Cambridge University Press, New York, 1961), Chap. 2 on the ellipse.

[14] William C. Parke and O. Bergmann, Amer. J. Phys. **35**, 1131 (1967).

[15] James L. Cronin, Jr. and L. C. Jones, Amer. J. Phys. **36**, 758 (1968). Uses Varignon's theorem.

[16] Sir Harrie Massey, *Space Physics* (Cambridge University Press, New York, 1964).

[17] R. Weinstock, Amer. J. Phys. **30**, 813 (1962).

[18] Seville Chapman, Amer. J. Phys. **31**, 213 (1963).

[19] The basic idea is not new. Of these authors only Cronin and Jones give a reason why a $1/r^2$ force implies a $1/r$ energy relationship, making use of Varignon's theorem. In Appendix A. of this paper, only simple high-school physics, algebra, and geometry are needed to show this essential relationship.

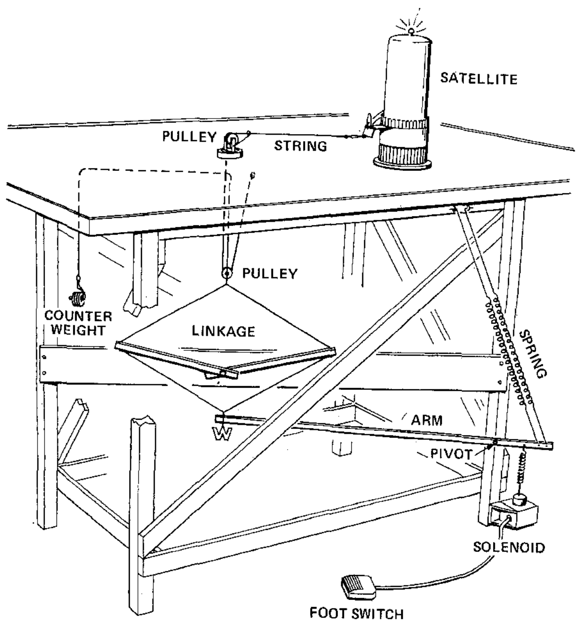


FIG. 2. Kepler's laws demonstration apparatus. The W hanging on the bottom of the Peaucellier linkage shows where a constant force should be applied.

Of course, one really needs to show that this equation for the ellipse yields the same orbit as the more familiar definitions (the locus of points for which the sum of the distances to two foci is constant), or equations (e.g., the standard form for the "stretched circle" $x^2/a^2+y^2/b^2=1$). (See Appendices A and B.)

KEPLER'S LAWS

Kepler's first law, that the orbit is an ellipse with force center at one focus, has been established.

Equation (4) really establishes the second law of equal areas being swept out in equal times. If in Fig. 1 the satellite is a point P and a moment later has moved to point P_1 in the time t_1, the time rate of sweeping out of area A is constant, thus

$$A/t=0.5qPP_1/t_1=0.5qv=K_2/m=\text{const} \quad (7)$$

Since the area of a circle of radius b is πb^2, the area of an ellipse, with all abscissae of the circle increased in the ratio (a/b), is $\pi b^2(a/b)=\pi ab$. The average rate of sweeping out of area by the radius to the planet is $\pi ab/T$, where T is the period. Thus

$$K_2/2m=\pi ab/T. \quad (8)$$

Compare Eqs. (5) and (6) obtaining

$$b^2=0.5K_2^2/K_1m, \qquad a=GMm/2K_1,$$

$$K_1=GMm/2a. \quad (9)$$

Incidentally, we should note that the magnitude of the total energy K_1 determines the semimajor axis a, while for a given total energy, the angular momentum determines the semiminor axis b.

Substitute Eq. (9) into Eq. (8), transpose and cancel, obtaining

$$T^2=4\pi^2a^3/GM, \quad (10)$$

which expresses Kepler's third law of the proportionality of T^2 and a^3, with the period independent of the minor axis.

It must not be supposed that Kepler reached his magnificent conclusions by any straightforward

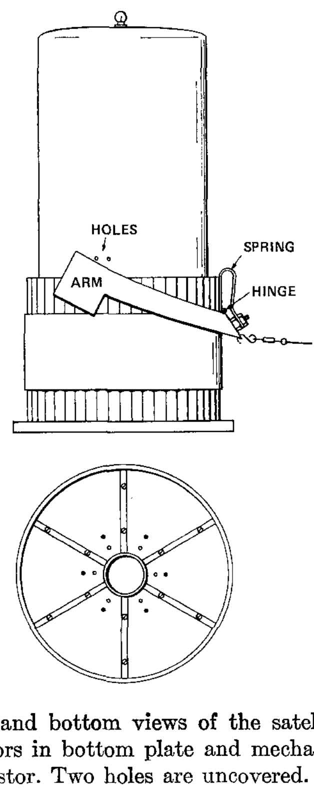
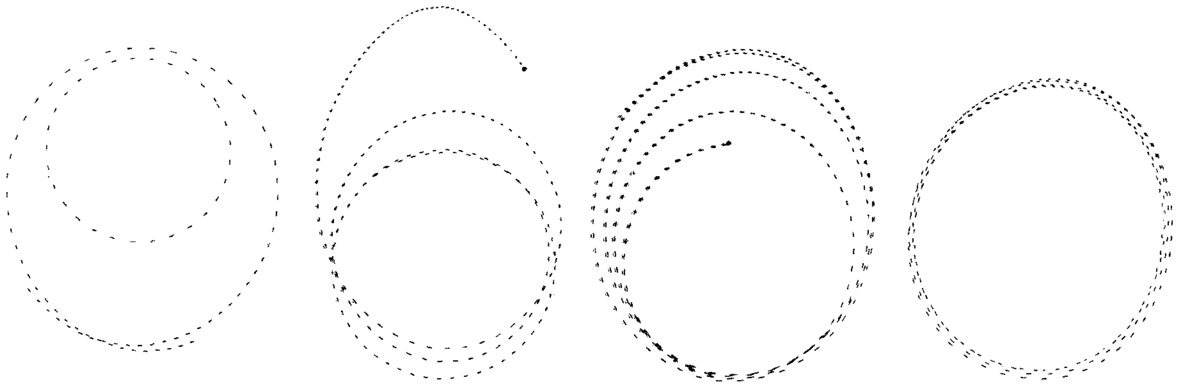


FIG. 3. Side and bottom views of the satellite showing recessed sectors in bottom plate and mechanism for the variable thrustor. Two holes are uncovered. Some of the thirty 1-cm diam, 10-cm-long iron rods used for adding mass to the satellite are indicated schematically.



(a) (b) (c) (d)

FIG. 4. Photographs of typical orbits. Orbits do not always close, and though not perfect, are close approximations of Kepler ellipses. (a) Two orbits; note that velocity in the ellipse near "perigee" exceeds that in the circle. (b) Decaying elliptic orbit; no energy added. (c) Expanding orbit with energy added; orbit becomes stable in size after four revolutions. (d) Several typical repetitive orbits.

rational process.[20–22] During about two decades he reached many conclusions, the majority of them trivial or wrong. Of some of the latter group he was most proud. Nevertheless, amongst the chaff were these gems of genius. Credit goes to Newton for grasping that, taken together, Kepler's laws implied a *universal* force of gravitation varying inversely with the square of the distance.

THE APPARATUS

The apparatus is shown in Figs. 2 and 3. The three main elements are: (1) a 1.2×1.5-m level table on which (2) an air-supported "satellite," made from a small vacuum cleaner is constrained to move in an elliptic orbit by an inverse-square central force generated by (3) a Peaucellier linkage and a long spring under the table. The position of a small flashlight bulb on top of the satellite has been photographed (to a scale of 1/15.0) by a Polaroid camera equipped with a rotating shutter making 10 exposures/sec, so that the orbit can be studied and measured.

[20] Arthur Koestler, *The Watershed: A Biography of Johannes Kepler* (Doubleday & Co., Inc., New York, 1960). See also Refs. 21 and 22.

[21] Florian Cajori, *Sir Isaac Newton's Mathematical Principles* (University of California Press, Berkeley, 1934), pp. 56–57. Note especially "Proposition XI, Problem VI. If a body revolves in an ellipse it is required to find the law of centripetal force tending to the focus of the ellipse."

[22] Alfred M. Bork, Amer. J. Phys. **35**, 342 (1967).

Eccentric orbits are fairly good ellipses (actually elliptical spirals, degenerating to nearly circular spirals as the system gradually loses energy), with the focus of the ellipse at the center of force. The velocity is seen to be greater at smaller radii. The squares of the period are measured to be proportional to the cubes of the major dimension of the orbit. Various orbits are shown in Figs. 4–6.

After brief instruction, anyone of modest dexterity can inject the satellite into a reasonable orbit. To duplicate orbits on successive trials, a "launcher," something like an adjustable slingshot, is useful.

The "planet" or satellite is made from a $9.95 Sears portable-vacuum cleaner (no longer listed in the catalogue), which generates a differential pressure of 50 cm of water. It is used as a blower,

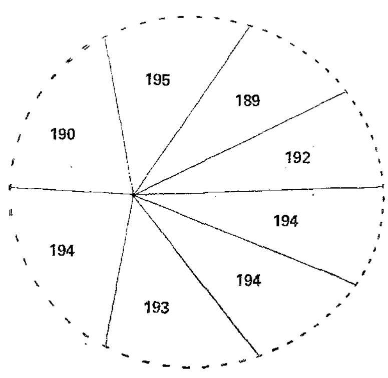


FIG. 5. Elliptic orbit showing the law of equal areas. Numbers show area in square millimeters counted on the original photograph.

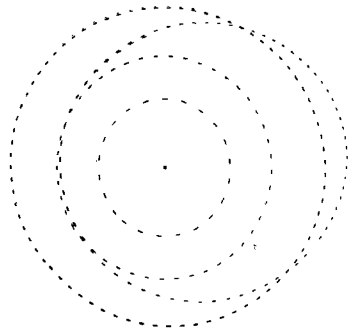


FIG. 6. Circular and elliptic orbits showing that period, T, is proportional to $a^{1.5}$. Respectively, the periods are 2.16, 4.5, 6.8 (ellipse), and 7.67 sec. The semimajor axes, a, are 0.167, 0.273, 0.360, 0.386 m. The ratios $T/a^{1.5}$ are 1003, 994, 991, 1020. Reductions of about five percent in the ratios are to be expected if significant increases are made in the amount of thrust, e.g., from none to the best amount to twice as much (which makes all orbits rapidly increase in size).

with air entering a lower plenum covered by a $\frac{1}{4}$-in. (about 6.3 mm) aluminum plate 14.0 cm in diam, through which six pairs of No. 41 holes are bored (diam 2.44 mm) to admit air to six recessed pie-shaped 60° sectors in the bottom of the plate. Air escapes under the narrow rim. If the satellite tips slightly in one direction, partially blocking air from escaping from that sector, the pressure rises in that sector, restoring the satellite to its normal vertical direction, with positive stability. With only a single air chamber under the plate, the planet is unstable and wobbles. Similar principles are used in commercial air-supported vehicles. The satellite, which weighs from 2100 to 3750 g depending on the loading (which must be uniform if escaping air is to create no net thrust), lifts off the table with about 55 to 80 V applied to its 115-V universal motor, but it is usually operated with normal voltage. It rides about 0.005 to 0.01 cm above the table. Power is supplied through a very flexible lightweight wire, Belden 8430 2-32(7×40) twisted-pair phono cable, which hangs down from above.

The table top is made of $\frac{3}{4}$-in. (approximately 2 cm) Novaply, a very smooth board of pressed wood chips. It is imperative that the top be a plane surface, with no sag, clean, and level. The coefficient of friction between the satellite and table at slow speeds is measured to be approximately 0.0005 (e.g., a deviation of 0.5 mm in level between supports 1 m apart is observable).

The satellite is pulled toward the center by a string that goes over a horizontal-axis, lightweight, 6-g, ball-bearing pulley held in a fork that rotates in ball bearings about a vertical axis. The string continues down through a hole in the fork and then through the table. Instead of tying the string to the satellite, for a while I tied it to a small pulley held in place by a single loop of thread, whose length was somewhat greater than the circumference of the satellite, and which went around the satellite. Thus the string did not exert any significant torque on the satellite even when the satellite was spinning on its axis, which it seldom did, except with quite eccentric orbits. This refinement was of marginal value, and after I added some thrust, which is essential to compensate for angular-momentum losses (see below), it was abandoned.

Under the table the string from the satellite goes around a lightweight, 7-g, precision-ball-bearing, movable pulley that serves in the usual way to change the force and motion by factors of two. The other end of the string is tied to the bottom of the table. Both ends of the string have small metal swivels to relieve twisting of the string. The swivels were obtained from a fisherman's sporting-goods store. A woven string seems to have advantages over a twisted string, or even ball chain.

The lightweight pulleys mentioned in the two preceding paragraphs are crucial, and must be of

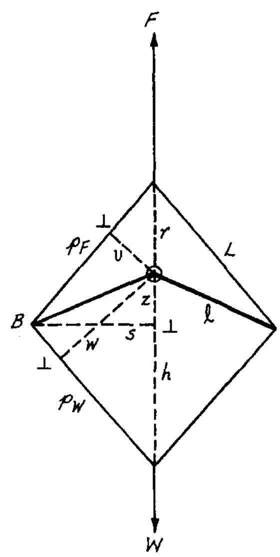


FIG. 7. Peaucellier linkage with four equal tension links L, and two equal compression links l, pivoted to the frame in the center.

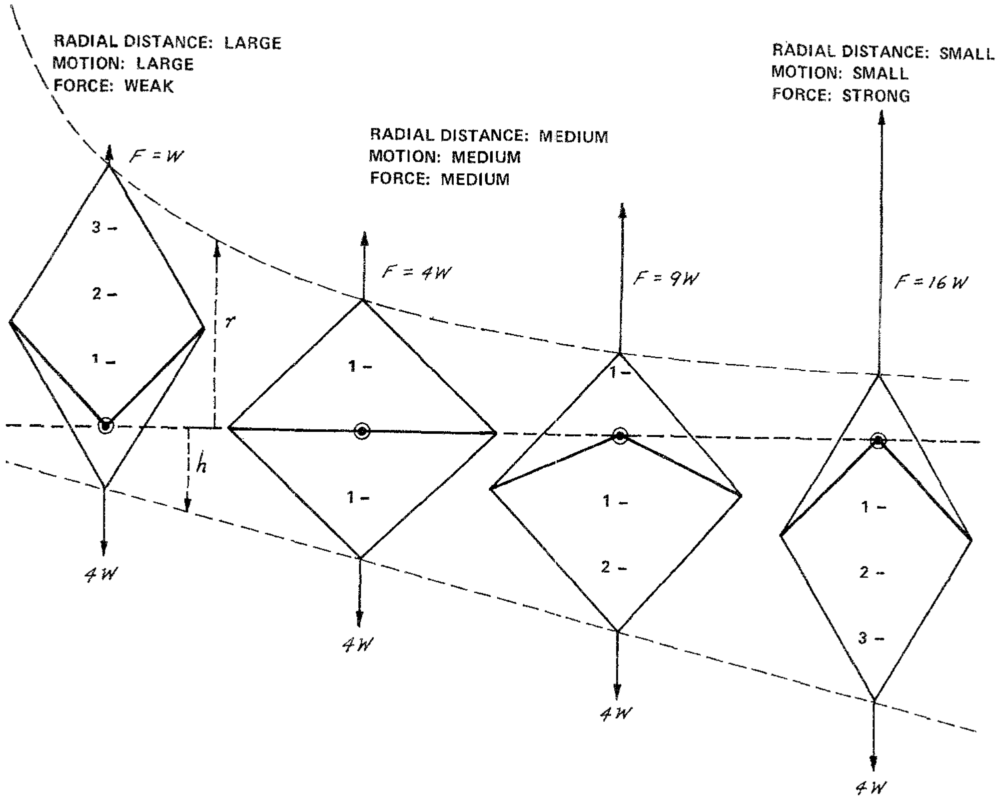


FIG. 8. Peaucellier linkage in several positions showing that constant linear motion of a force $4W$ on the bottom, varying the input distance h, results in an inverse non-linear-motion of the output distance r, such that the product hr is constant, and also results in a force F varying inversely as r^2.

high quality or the friction will result in all elliptical orbits being changed to nearly circular orbits in less than a single orbit. I experienced considerable difficulty with these pulleys, and can only say that not all ball bearings are alike. With care, a counterbalanced weight can be made to go up or down by adding or subtracting a differential force of about 2% of one of the weights. The upper pulley, which is held in a fork, of course must turn in azimuth as the satellite goes around in orbit. I had no trouble with that motion. The bearings are held in a heavy aluminum cylindrical disk screwed to the table.

The key to the inverse-square force law is a linkage, invented by Peaucellier in 1864, which is one of the most famous linkages, and the first to be used in constructing a straight-line linkage. There was a tremendous interest in linkages in the 1870's which was followed by almost a complete hiatus until the decade from about 1940–1950, when it was recognized that they could be used for computing. The subsequent rise of the digital computer eliminated that application.[23–33] The Peaucellier cell is a six-bar linkage (see Figs. 7 and 8). It may appear in a variety of forms generally with four equal links and two equal

[23] J. D. C. de Roos, *Linkages* (D. Van Nostrand Co., Inc., New York, 1879).

[24] A. Svoboda, *Computing Mechanisms and Linkages* (McGraw-Hill Book Co., New York, 1948), Chap. 2 on bar-linkage computers.

[25] Peter Schwamb and Merrill James, *Elements of Mechanism* (John Wiley & Sons, Inc., New York, 1947), 6th ed., rev. pp. 139–145. Especially these pages on linkages.

[26] Joseph Stiles Beggs, *Mechanism* (McGraw-Hill Book Co., New York, 1955), pp. 202–207. Especially these pages on linkages.

[27] "Linkages," Encyclopedia Britannica **14**, 163 (1943).

[28] B. E. Meserve, Math. Teach. **39**, 372 (1946).

[29] J. A. Hrones and G. L. Nelson, *Analysis of the Four Bar Linkage* (John Wiley & Sons, Inc., New York, 1951).

[30] E. W. Pike, T. R. Silverberg, and P. T. Nickson, Machine Design **23**, 105 (1951).

[31] H. G. Conway, Machine Design **22**, No. 1, 90 (1950).

[32] Leo S. Packer, Machine Design **25**, No. 9, 171 (1953).

[33] B. J. Bedell, Space Flight **5**, No. 6, 190 (1963).

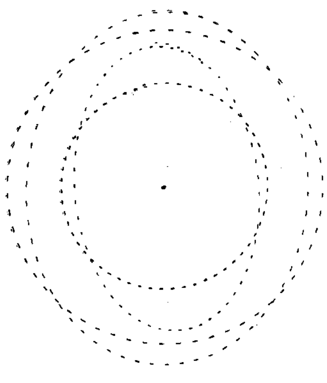


FIG. 9. Elliptical and circular orbits generated by a Hooke's-law spring with central force proportional to the first power of the distance. Note that for any one orbit the law of equal areas in equal times applies, that orbits are ellipses, but that the center of force is at the *center* of the ellipse, and that the period is independent of orbit size (all four orbits have periods of 5.6–5.7 sec).

links of different length. In this arrangement, which is less common than some others, the four links in tension are 0.785-mm diam steel piano wires and the two links in compression are aluminum strips. Most linkage analyses are concerned with kinematics (amounts of motion for various input displacements); in this apparatus we are concerned with dynamics (forces). The characteristic of a Peaucellier cell invariably discussed is that it is an inversor, that is, if h and r are the translations of the input and output, $h=(L^2-l^2)/r$, where L^2-l^2 is a constant of the apparatus, L and l being the lengths of the links. For the present linkage $L^2-l^2=(14.0\text{ cm})^2$. Although I have examined many dozens of papers and books on linkages, I have not found any statement that the Peaucellier cell transforms a constant force into an exact inverse-square force (as shown in Appendix A), though it seems impossible that no one has known it before. Thus a constant force (a weight) on the bottom of the linkage creates an inverse-square force in the string that goes to the satellite. A person who tries pulling on the satellite for the first time (being used to years of Hooke's law, linear-force relations) is *invariably amazed* at the feel of an inverse-square law force that gets weaker as the satellite is pulled farther out.

Actually, to reduce inertia forces when the satellite is in an elliptical orbit, I have substituted for the weight a long lever arm and two long springs. From the point where an S hook connects the linkage to the arm, it is 68.0 cm to the pivot. Springs are connected at 11.4 and 14.0 cm from the pivot. Stretched lengths are greater than unstretched lengths by 25.5 and 29.3 cm, respectively. The arm weighs 170 g with most weight near the pivot. By fastening the springs to make the proper angle with the lever arm, the torque on the arm (68 cm×399 g) is constant within a percent over the motion of the linkage. The springs and arm have less mass than the earlier weight used, and yield better elliptic orbits. Ball bearings are used in the arm and in the compression members of the linkage, although they may not be necessary since the differential friction without them was only about 2%.

The linkage is not of zero weight, and a small 30-g counterweight is used, connected to a string that goes over two small pulleys, to balance the weight of the linkage and the movable pulley below the table.

Pedagogically, it may be better to show a constant weight on the bottom of the linkage than to use the lever arm and spring. The product of the weight and its change in height constitutes the change in potential energy of the planet as it moves in an elliptic orbit.

Over-all, the actual force on the planet is accurately an inverse-square force (a plot on log–log paper is straight within experimental error, and of slope −2.0), under conditions of balance. In the dynamic situation, there is obviously some friction so that the forces for a planet approaching the center are less than those when it is receding. The result is that eccentric orbits tend to become circular. Of course, there is a gradual loss of energy too. The friction losses in circular orbits are quite small, and depending on the initial conditions, the satellite will make dozens of circular orbits before completing its gradual spiral into the center.

By allowing a tiny amount of air to escape through a small hole in the side of the vacuum cleaner, one can add a small amount of thrust (a few grams), and even can get the satellite orbit to increase in size. Such extra force can compensate exactly for friction, but only at one radius.

To deal more effectively with the problems of energy and momentum losses, two additional devices were added. Either or both can be disabled when desired.

First, a variable thrustor was added to the satellite. A spring linkage was connected to the string providing the inverse-square law force, so that the string force determined the position of a small movable arm. Originally, this arm gradually uncovered a set of up to ten 2.7-mm diam holes as the satellite approached the focus. This "blast of air" provided a thrust of from zero to approximately 28 g. It added energy and momentum to the system—unfortunately, too much of the latter in proportion to the former, so that the resulting trajectories tended more toward rosettes with a laterally displaced focus than to ellipses.

Subsequently, a new set of five 1.8-mm diam holes was used so that the satellite would make circular orbits of almost any size with only very slow changes in radius.

For elliptic orbits an energy adder was installed. It is a solenoid (Guardian 14AC 18 VBC 115-V 60 Hz) with a pull length up to 3.8 cm connected to a spring of force-constant 140 g/cm connected to the large horizontal lever arm, typically 7 cm from the pivot, so that the demonstrator using a foot switch can reduce the force on the satellite (by approximately 20%–10% depending on the distance from the focus), when it is receding. By modifying the inverse-square law slightly (adding a 2.5-cm short link in the string to an otherwise properly adjusted satellite) so that the force is a bit too great near the focus (the order of 10%), then the actual dynamic force including pulley and linkage friction is closely the same inverse-square law when the satellite is both receding with the solenoid on and approaching with it off. The combination of variable thrustor and energy adder balances both momentum and energy losses, and permits obtaining a dozen identical eccentric ellipses (± about 1 cm).

Some people are surprised by certain principles well illustrated by this apparatus, that for a satellite in orbit a continuing retarding force *increases* the speed as the satellite moves to a smaller orbit closer to the center of force. Likewise, continuing forward thrust *decreases* the speed as the satellite moves to a more distant orbit.

An impulse (i.e., an instantaneous force, or in any case, application of force for a time short compared to an orbital period) produces results more in accordance with ordinary intuition, that is, an appropriate abrupt change of velocity. An abrupt increase of velocity results in the satellite covering the immediate part of the orbit faster, but results in an orbit whose major axis and period are increased, so that overall the average speed is reduced. Thus in earth-satellite docking, to overtake in a time short compared to an orbital period, use positive thrust; to overtake in a time comparable with an orbital period, use negative thrust.

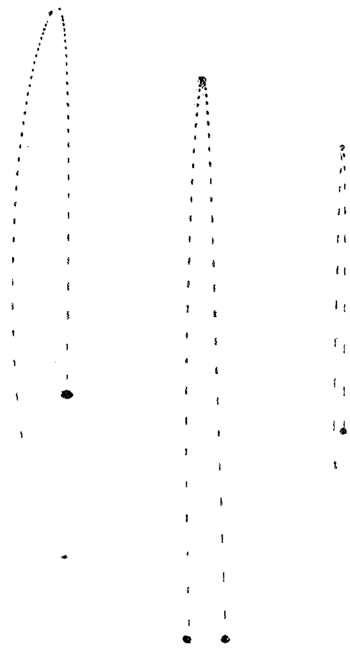


FIG. 10. Acceleration generated by (left) an inverse square-law force, (center) a constant force, (right) a first power force. Caution: determination of the force law by taking second differences of position data and smoothing can result in effectively canceling most of the data. (References 34 and 35.)

As stated previously, the present apparatus is just a feasibility model. There are many design optimizations that have not been investigated thoroughly (size of pulley, mass of satellite, size of table, dimensions of linkage in horizontal or vertical arrangement, quantity of air for support, type of string or chain, magnitude of force, magni-

[34] E. M. Pugh, Amer. Phys. Teach. 4, 70 (1936).

[35] Seville Chapman, *Laboratory Manual, Engineering Physics: Mechanics and Sound* (National Press Books, Palo Alto, Calif., 1947), Experiment 3.

tude and timing of compensating thrust or energy addition, automatic-magnetic or photoelectric control for the energy adder or for unwinding the power wire, costs and labor, etc.).

CONCLUSION

The device illustrates all of Kepler's laws and is particularly effective in showing the interchanges between potential and kinetic energy. Centrifugal force, statics, friction, energy losses, momentum losses—all are illustrated. Ellipses, whether from an inverse-square force law or (with minor changes) from a Hooke's-law force, can be compared and the differences noted (Fig. 9). Acceleration from an inverse-square force, from a constant force, or from a linear force can be photographed and measured (Fig. 10). The apparatus can be used for demonstration or laboratory experiment. High-school algebra, geometry, and physics are sufficient prerequisites for a complete understanding of elliptic orbits.

APPENDIX A

Peaucellier Linkage Shows $1/r^2$ Force Yields $-1/r$ Energy

The Peaucellier linkage, Figs. 7, 8, and 11, is an inversor. It converts a constant force W to a force F proportional to $1/r^2$. The energy is shown to be proportional to $-1/r$.

From considerations involving right triangles

$$L^2=(r+z)^2+s^2 \qquad (A.1)$$

$$L^2=(h-z)^2+s^2 \qquad (A.2)$$

$$l^2=s^2+z^2$$

or,

$$s^2=l^2-z^2. \qquad (A.3)$$

Substitute Eq. (3) in Eqs. (1) and (2):

$$L^2-l^2=r^2+2rz \qquad (A.4)$$

$$L^2-l^2=h^2-2hz. \qquad (A.5)$$

Subtract Eq. (4) from Eq. (5):

$$h^2-r^2=2z(h+r). \qquad (A.6)$$

Factor and cancel:

$$2z=h-r. \qquad (A.7)$$

Substitute Eq. (7) in Eq. (4):

$$L^2-l^2=r^2+hr-r^2 \qquad (A.8)$$

$$h=(L^2-l^2)/r. \qquad (A.9)$$

Equation (9) is the kinematic relationship showing the well-known result that the Peaucellier cell is an inversor, that is, that h is inversely proportional to r, since L^2-l^2 is a constant of the apparatus.

From symmetry, considering the tensions p in links L,

$$F/W=p_F/p_W. \qquad (A.10)$$

Point B is in equilibrium, hence torques at B about the center must balance, thus

$$p_F u=p_W w$$

or

$$p_F/p_W=w/u. \qquad (A.11)$$

From similar triangles

$$w/u=h/r. \qquad (A.12)$$

Hence

$$F/W=h/r. \qquad (A.13)$$

Substitute for h from Eq. (9):

$$F/W=(L^2-l^2)/r^2$$

or

$$F=(L^2-l^2)W/r^2, \qquad (A.14)$$

showing that for a constant weight W, the force F

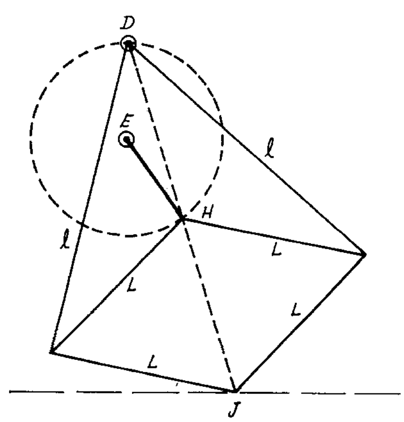


Fig. 11. Peaucellier straight-line linkage. There are two fixed pivots D and E. Link $EH=DE$. Point J moves in an exact straight line. If D and E are vertical, J moves horizontally (and a weight may be hung from J with no "pendulum swing").

is an inverse-square law, so that this linkage is useful for our purpose.

The work done by a force F in moving an object from r_2 or r_1 is the energy yielded by W in moving from h_2 to h_1, thus

$$\text{Work} = W(h_2 - h_1). \qquad \text{(A.15)}$$

Substitute for h from Eq. (9):

$$\text{Work} = (L^2 - l^2)W(1/r_2 - 1/r_1). \qquad \text{(A.16)}$$

If r_2 were infinity, the work done moving to r_1 would be

$$\text{Work} = (L^2 - l^2)W(-1/r_1), \qquad \text{(A.17)}$$

so that if the energy is regarded as zero at infinity, the energy of the object at a distance r from the center of force is

$$\text{Energy} = -(L^2 - l^2)W/r, \qquad \text{(A.18)}$$

proving that an inverse-square law force, yields an inverse-first-power energy relationship. No calculus has been used.

APPENDIX B

Equivalence of Different Equations for the Ellipse

In Fig. 12 with O as center, draw a circle of radius b. For any point on the circle having coordinates (X, y), clearly

$$X^2 + y^2 = b^2,$$

or the circle is defined by

$$X^2/b^2 + y^2/b^2 = 1.$$

Now keep all ordinates the same for a new curve so that $y = y$, but extend all abscissae in the ratio a/b; that is, in place of X put

$$x = (a/b)X \quad \text{or} \quad X = (b/a)x.$$

Substitute for X obtaining

$$x^2/a^2 + y^2/b^2 = 1$$

as the standard, well-known equation defining the new curve, the "stretched circle" or ellipse.

This same equation for the ellipse results from the definition that the sum of the focal radii for any point $P(x, y)$ on the ellipse is a constant, $2a$. Thus

$$[(x+c)^2 + y^2]^{0.5} + [(x-c)^2 + y^2]^{0.5} = 2a.$$

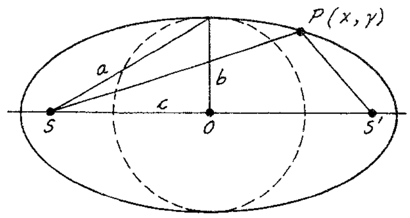


FIG. 12. The ellipse. For any point P, $SP + PS' = 2a$.

Transpose one radical, square both sides, simplify, leave only the remaining radical on one side, square again, simplify, simplify further noting that $a^2 - c^2 = b^2$, and the expression reduces to the standard form for the ellipse

$$x^2/a^2 + y^2/b^2 = 1.$$

Appendix C shows that the pedal form follows from the above definitions. Thus all four definitions of the ellipse are equivalent: the stretched circle, the sum of the focal radii $= 2a$, the standard form $x^2/a^2 + y^2/b^2 = 1$, and the pedal form.

APPENDIX C

To Derive the Pedal Equation for the Ellipse

This derivation is not commonly available in a form readily intelligible to students, (I know of no place), so it is summarized below. (See Fig. 13.)

(1) Choose a point S, and lay off an arbitrary distance $2a$ to a point Q. It will turn out that S is the focus of the ellipse and $2a$ is equal in length to the major axis.

(2) Choose a point S' nearer to S than Q is to S.

(3) Bisect SS' at O.

(4) Draw a circle of radius a about O.

(5) Draw QS' intersecting the circle at Y.

(6) Since $SQ = 2OY$ and $SS' = 2OS'$, therefore OY is parallel to SQ, and Y bisects QS'.

(7) Draw VY perpendicular to $S'Q$, intersecting SQ at P.

(8) Since $PS' = PQ$, therefore $SP + PS' = 2a$. This equation often is used to define an ellipse as that locus of points P, the sum of whose distances from two foci (S and S') is constant. Thus point P is on the ellipse.

(9) Since O is the midpoint of SS' and RY, therefore RS and $S'Y$ are parallel and equal. Extend RS toward V.

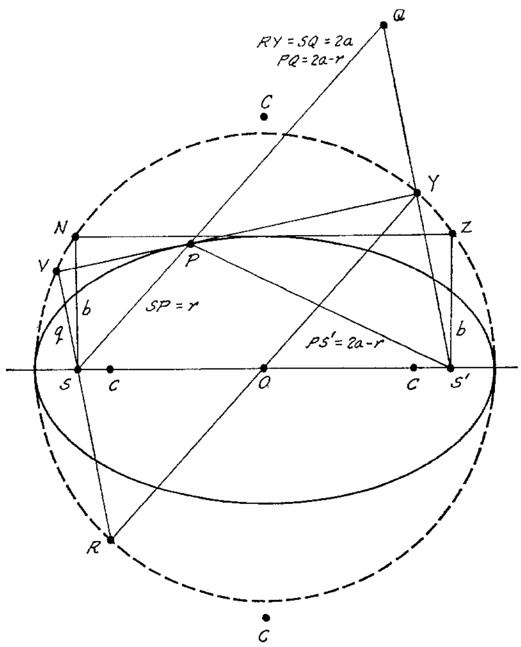


FIG. 13. Diagram for the pedal equation. The pedal distance is SV or q. Centers of curvature for the extremities of the ellipse are at points C, at distances b^2/a or a^2/b from the ellipse.

(10) Since RS is parallel to $S'Y$, RS is perpendicular to VY, and hence RS intersects VY at V because the right angle at V subtends the diameter of the circle.

(11) Draw SN and $S'Z$ perpendicular to SS'. SN and $S'Z$ are both equal to the semiminor axis b, since $b^2 = a^2 - (SO)^2$.

(12) Consider triangles SVN and $S'ZY$. The angles at S and S' are equal since their sides are parallel, and the angles at V and Z are equal since each is the sum of 90° plus one-half arc NY. Hence the triangles are similar, and calling $SV = q$, then $q/b = b/S'Y$.

(13) Consider triangles SVP and $S'YP$, the latter of which is equal to triangle QYP. The triangles are similar, therefore $r/q = (2a-r)/S'Y$.

(14) Eliminate $S'Y$ from the two equations obtaining:

$$b^2/q^2 - 2a/r = -1,$$

which is the pedal equation of the ellipse, for the locus of points P.

(15) It remains to show that VY is actually tangent to the ellipse at P. The two radii SP and $S'P$ make equal angles with VY at P. To show that they make equal angles with the tangent at P, draw radii to an adjacent point P'. The increase in length of one radius must be equal to the decrease in length of the other radius (see 8 above). Extend the radii to form a small rhombus near P. When P is very near to P', the diagonal PP' of the rhombus coincides very nearly with the tangent at P. The angles made by the radii (which are sides of the rhombus) with the diagonal (which in the limit is the tangent) are equal. Hence VY is the tangent to the ellipse, and the pedal (defined as the perpendicular to the tangent) is q.

This derivation of the pedal equation is closely related to the demonstration where an arbitrary point S' is picked in a circular sheet of wax paper (center at S) which is then folded many times (along lines like VY) to bring a series of points Q at the edge of the sheet into coincidence with S'. The resulting pattern of creases identifies an ellipse.