

Sum of angular momenta [CT X; G 4.4.3]

I - Physical motivation

In CM \exists several important situations in which interacting physical systems exchange angular momentum, but where the total ang. mom. of the system is conserved.

This remains true in QM, but with several additional and quite subtle features

Examples

1) 2-particle interaction that can be described by a potential energy of the form $V(|\vec{r}_1 - \vec{r}_2|) = V(r)$ ["central potential", or "radial force"]

$$H = H_1 + H_2 + V(r)$$

Assume $[L_j, H_j] = 0$ (each isolated system conserves its AM)

$$\left[\text{eg. } H_j = \frac{1}{2m_j} \vec{P}_j^2 + V(\vec{r}_j) \right]$$

Individual AM's are not conserved under H :

$$\text{ex: } [L_{jz}, H] = [L_{jz}, V] = [X_j P_{jy} - Y_j P_{jx}, V] = V(r) (Y_j P_{jx} - X_j P_{jy}) + (X_j P_{jz} - Y_j P_{jx}) V/r$$

$$\text{in Cartesian coords: } = -i\hbar \left[V(r) \left(Y_j \frac{\partial}{\partial x_j} - X_j \frac{\partial}{\partial y_j} \right) + X_j \frac{\partial}{\partial y_j} [V \cdot] - Y_j \frac{\partial}{\partial x_j} [V \cdot] \right]$$

$$= -i\hbar \left[V(r) \left(y_j \cancel{\frac{\partial}{\partial x_j}} - x_j \cancel{\frac{\partial}{\partial y_j}} \right) + V(r) \left(x_j \cancel{\frac{\partial}{\partial y_j}} - y_j \cancel{\frac{\partial}{\partial x_j}} \right) + x_j \frac{\partial V}{\partial y_j} - y_j \frac{\partial V}{\partial x_j} \right]$$

$$= -i\hbar \left(x_j \frac{\partial V}{\partial y_j} - y_j \frac{\partial V}{\partial x_j} \right)$$

now: $r = \left(r_1^2 + r_2^2 - 2\vec{r}_1 \cdot \vec{r}_2 \right)^{1/2} \Rightarrow \frac{\partial V}{\partial y_j} = \frac{\partial V}{\partial r} \frac{\partial r}{\partial y_j} = \frac{\partial V}{\partial r} \cdot \frac{1}{2r} (-2y_j) = -\frac{1}{r} \frac{\partial V}{\partial r} y_j$

where $j = \begin{cases} 1 & \text{if } j=2 \\ 2 & \text{if } j=1 \end{cases}$ (complement)

However, the L_z AM $\vec{L} \equiv \vec{L}_1 + \vec{L}_2$ is conserved:

$$[L_z, H] = [L_z, V] = [L_{1z}, V] + [L_{2z}, V] = -i\hbar \left[\frac{\partial V}{\partial r} (x_2 y_1 - y_2 x_1) + \frac{\partial V}{\partial r} (x_1 y_2 - x_2 y_1) \right] = 0$$

(by symmetry, the same is true of L_x and L_y)

Alternatively: in spherical coordinates (not immediately simpler)

$$[L_{jz}, V] = i\hbar \frac{\partial}{\partial \varphi_j} (V \cdot) - i\hbar V \frac{\partial}{\partial \varphi_j} = i\hbar \frac{\partial V}{\partial \varphi_j} = i\hbar \frac{\partial V}{\partial r} \frac{\partial r}{\partial \varphi_j}$$

but $\frac{\partial r}{\partial \varphi_j} = \frac{1}{2r} \frac{\partial}{\partial \varphi_j} \left(\cancel{r_1^2} + \cancel{r_2^2} - 2\vec{r}_1 \cdot \vec{r}_2 \right) = \frac{-1}{r} \frac{\partial}{\partial \varphi_j} \left[r_1 r_2 \sin \theta_1 \sin \theta_2 (\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2) + r_1 r_2 \cos \theta_1 \cos \theta_2 \right]$

$$= -\frac{r_1 r_2}{r} \sin \theta_1 \sin \theta_2 \left[\cos \varphi_j \sin \varphi_j - \sin \varphi_j \cos \varphi_j \right]$$

$$\rightarrow [L_{1z} + L_{2z}, V] = i\hbar \frac{\partial V}{\partial r} \left(\frac{\partial r}{\partial \varphi_1} + \frac{\partial r}{\partial \varphi_2} \right) = -\frac{r_1 r_2}{r} \sin \theta_1 \sin \theta_2 \left[c \cancel{\varphi_1} s \cancel{\varphi_2} - s \cancel{\varphi_1} c \cancel{\varphi_2} + c \cancel{\varphi_2} s \varphi_1 - s \cancel{\varphi_2} c \varphi_1 \right] = 0$$

ii) Interaction of the form $H_{so} = V(r) \vec{L} \cdot \vec{S}$ ["spin-orbit" interaction]

[between "orbital" and spin AM variables of the same particle]

$$\text{In this case e.g. } [L_z, H_{so}] = V(r) [L_z, L_x S_x + L_y S_y + L_z S_z]$$

$$= V(r) ([L_z, L_x] S_x + [L_z, L_y] S_y) = i\hbar V(r) [L_x S_y - L_y S_x]$$

$$\text{and similarly } [S_z, H_{so}] = i\hbar V(r) [S_x L_y - S_y L_x]$$

→ defining "total AM" $\vec{J} \equiv \vec{L} + \vec{S}$, we have $[J_z, H_{so}] = 0$

Again, by rotational symmetry of H_{so} , same is true of $J_x, J_y \rightarrow [\vec{J}, H_{so}] = 0$

Other examples (exercise)

Ising Hamiltonian $K S_1 z S_2 z$

Zeeman Ham.: $\mu (\vec{L} + 2\vec{S})$

Dipole-dipole Ham.: $k \cdot \frac{1}{r^3} (3(\vec{S}_1 \cdot \hat{r})(\vec{S}_2 \cdot \hat{r}) - \vec{S}_1 \cdot \vec{S}_2)$

II - Commutation relations and state space

In general: define "total angular momentum" as a sum $\vec{J} = \vec{J}_1 + \vec{J}_2$, where \vec{J}_1, \vec{J}_2 are any kind of vector operators whose components obey the A.M. commutation relations (ie: \vec{J}_i can be orbital AM, spin AM, or even something else).

Clearly, \vec{J} itself also obeys these relations, eg

$$[J_x, J_y] = [J_{1x} + J_{2x}, J_{1y} + J_{2y}] = [J_{1x}, J_{1y}] + [J_{2x}, J_{2y}] = i\hbar(J_{1z} + J_{2z})$$

valid since operators on subsystem "1"
commute w/ ops on subsystem "2"

$$= i\hbar J_z$$

⇒ can use machinery of AM algebra

In particular: $[J^2, J_z] = 0$, and it must be possible to construct common eigenvectors $|j, m\rangle$ satisfying

$$J^2 |j, m\rangle = j(j+1)\hbar^2 ; \quad J_z |j, m\rangle$$

where j is some integer or half-integer, and $m \in \{-j, -j+1, \dots, j-1, j\}$

However: since \vec{J}_1, \vec{J}_2 act on different Hilbert spaces \mathcal{E}_1 and \mathcal{E}_2 , then

\vec{J} is actually an operator on $\mathcal{E}_1 \otimes \mathcal{E}_2$!

→ $|j, m\rangle$ are states of $\mathcal{E}_1 \otimes \mathcal{E}_2$!

Central Q: what are these states $|j, m\rangle$? In particular, how are they related to the "product states" $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$?

To answer: need to investigate the algebraic relations between the total AM operators and the local ones

Some useful general relations

- i) $[J_z, J_{1z}] = [J_z, J_{2z}] = 0$ } [note: by symmetry also $[J_y, J_{1y}] = 0$]
- (obvious)
- ii) $[J_z, J_1^2] = [J_z, J_2^2] = 0$ } [note: by symmetry also $[J_y, J_1^2] = 0$]
- iii) $[J^2, J_1^2] = [J^2, J_2^2] = 0$ (proof: $[J^2, J_1^2] = [J_x^2, J^2] + [J_y^2, J^2] + [J_z^2, J^2] = 0$)
- iv) $[J^2, J_{1z}] \neq 0!$ Specifically

$$[J^2, J_{1z}] = 2ih(J_{1x}J_{2y} - J_{1y}J_{2x}) \quad (\text{and analogously for } [J^2, J_{2z}])$$

Prof: note $J^2 = (\vec{J}_1 + \vec{J}_2) \cdot (\vec{J}_1 + \vec{J}_2) = J_1^2 + J_2^2 + 2\vec{J}_1 \cdot \vec{J}_2$

$$\begin{aligned} \rightarrow [J^2, J_{1z}] &= [J_1^2, J_{1z}] + [J_2^2, J_{1z}] + 2[\vec{J}_1 \cdot \vec{J}_2, J_{1z}] \\ &= 2[J_{1x}J_{2x} + J_{1y}J_{2y}, J_{1z}] = 2\overbrace{[J_{1x}, J_{1z}]}^{+ihJ_{1y}} J_{2x} + 2\overbrace{[J_{1y}, J_{1z}]}^{-ihJ_{1x}} J_{2y} \\ &\quad (\text{note: } [J^2, \vec{J}] = 2ih\vec{J}_1 \times \vec{J}_2) \quad (\text{cross product}) \end{aligned}$$

→ in general, states $|j, m\rangle$, with a well-defined value of j will not have well-defined values of m_1 or m_2 (ie, will be superpositions of states with different values of m_1 and m_2 !)

Main tool we will use to understand this: raising / lowering operators!

$$\text{Recall } L_{\pm} = L_x \pm iL_y$$

$$\text{Properties : } L_{\pm} |l, m\rangle = \sqrt{l(l+1) - m(m\pm 1)} \hbar |l, m\pm 1\rangle \Rightarrow L_+ |l, l\rangle = L_- |l, -l\rangle = 0$$

$$L_{\underline{-}} L_{\underline{+}} = L_x^2 + L_y^2 - i(L_y L_x - L_x L_y) = L_x^2 + L_y^2 \cancel{\pm \hbar L_z} = L^2 - L_z^2 \cancel{\pm \hbar L_z}$$

$$\rightarrow L_{\underline{-}} L_{\underline{+}} = L^2 - L_z^2 \cancel{\pm \hbar L_z}$$

$$\text{Note now : if } \vec{J} = \vec{J}_1 + \vec{J}_2$$

$$V) \text{ Useful identity : } J_{1x} J_{2x} + J_{1y} J_{2y} = \frac{1}{2} (J_{1+} J_{2-} + J_{1-} J_{2+})$$

$$\text{Proof : } J_{1+} J_{2-} + J_{1-} J_{2+} = (J_{1x} + iJ_{1y})(J_{2x} - iJ_{2y}) = 2(J_{1x} J_{2x} + J_{1y} J_{2y}) \\ + (J_{1x} - iJ_{1y})(J_{2x} + iJ_{2y})$$

$$\Leftrightarrow \vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2} (J_{1+} J_{2-} + J_{1-} J_{2+}) + J_{1z} J_{2z}$$

alternative form

III - Simplest case : sum of 2 spin $\frac{1}{2}$ AM

This special case allows us to introduce the appropriate mathematical technique and point out features of the result w/out getting too bogged down

Physically : describes f. example the "total spin" of system of 2 spin $\frac{1}{2}$ particles , such as a proton + electron (H atom!)

i) Notation since $s_1 = s_2 = \frac{1}{2}$ (fixed) we omit this, eg we simply write

$$|s_1 = \frac{1}{2}, m_{s_1}\rangle \otimes |s_2 = \frac{1}{2}, m_{s_2}\rangle \equiv |m_1, m_2\rangle \quad (\text{where } m_j \in \{-\frac{1}{2}, \frac{1}{2}\})$$

even more simply, we use the $| \pm \rangle$ notation (eg $|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\rangle \equiv |++\rangle$)

so the state space $\mathcal{E}_1 \otimes \mathcal{E}_2$ has basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$

Note

$$\begin{cases} S_i^2 |m_1, m_2\rangle = \frac{1}{2} \left(\frac{1}{2} + 1\right) \hbar^2 |m_1, m_2\rangle = \frac{3}{4} \hbar^2 |m_1, m_2\rangle, \forall m_1, m_2 \\ S_{jz} |m_1, m_2\rangle = m_j \hbar |m_1, m_2\rangle \end{cases}$$

In this case $\{S_{1z}, S_{2z}\}$ form a C.S.C.O.

total AM operator : $\vec{S} = \vec{S}_1 + \vec{S}_2$

Goal : construct an alternative basis $\{|S, M\rangle\}$ for this same Hilbert space, but satisfying

$$S^2 |S, M\rangle = S(S+1)\hbar^2 |S, M\rangle$$

$$S_z |S, M\rangle = M\hbar |S, M\rangle$$

Step 1 : classify states $|m_1, m_2\rangle$ according to M

- due to relation (11), $|m_1, m_2\rangle$ are automatically eigenstates of S_z :

$$S_z |m_1, m_2\rangle = (m_1 + m_2)\hbar |m_1, m_2\rangle \quad \rightarrow \quad M = m_1 + m_2$$

possible eigenvalues :
$$\left\{ \begin{array}{l} M = 1 \quad (\text{if } m_1 = m_2 = \frac{1}{2}) \\ M = -1 \quad (\text{if } m_1 = m_2 = -\frac{1}{2}) \\ M = 0 \quad (\text{if } m_1 = \frac{1}{2}, m_2 = -\frac{1}{2} \text{ or } m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}!) \end{array} \right.$$

Seems to correspond with a spin 1 ... however...

\rightarrow 2 different ways of getting $S_z = 0$! (?!) Why?

Matrix of S_z in this basis : $\hbar \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (note degeneracy)

step 2 : look at effect of S^2 within each degenerate subspace of S_z

Since $[S^2, S_z] = 0$, we know that S^2 must be block-diagonal w/ respect to the degenerate subspaces of S_z , ie, the matrix of S^2 in this basis must have the form

$$S^2 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b_1 & b_2 & 0 \\ 0 & b_2 & b_3 & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$$

Furthermore, for $M = \pm \hbar$, we can already guess that $S=1 \Rightarrow a=c=1/2(1+1)\hbar^2 = 2\hbar^2$

Indeed, using (v) :

$$S^2 |m_1, m_2\rangle = \left(S_1^2 + S_2^2 + \underbrace{2S_{1z}S_{2z}}_{2\vec{S}_1 \cdot \vec{S}_2} + S_{1+}S_{2-} + S_{1-}S_{2+} \right) |m_1, m_2\rangle$$

$$\text{for } m_1 = m_2 = \frac{1}{2} : S^2 |++\rangle = \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + \frac{2}{4}\hbar^2 \right) |++\rangle + 0 + 0 = 2\hbar^2 |++\rangle$$

$$\Rightarrow \text{in fact } |++\rangle = |S=1, M=1\rangle \text{ and similarly } |--\rangle = |S=1, M=-1\rangle$$

However S^2 cannot be diagonal in the degenerate space, because $[S^2, S_{1z}] \neq 0$

Step 3: find eigenstates within each degenerate subspace of S_z

2 possible techniques:

i) Find matrix elements b_1, b_2, b_3 , then diagonalize "B" block

$$\begin{aligned} S^z |+-\rangle &= \underbrace{\left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 - \frac{2}{4}\hbar^2\right)}_{\hbar^2 |+-\rangle} |+-\rangle + S_{1+} S_{2-} |+-\rangle + \underbrace{S_{1-} S_{2+} |+-\rangle}_{= S_- |+\rangle \otimes S_+ |- \rangle} = (iv) \\ &= \sqrt{\frac{3}{4} - \frac{1}{2}\left(-\frac{1}{2}\right)} \cdot \sqrt{\frac{3}{4} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} |\hbar^2 -+\rangle = \hbar^2 |-+\rangle \end{aligned}$$

$$\therefore S^2 |+-\rangle = \hbar^2 [|+-\rangle + |-+\rangle]$$

$$\text{similarly: } S^2 |-+\rangle = \hbar^2 [|-+\rangle + |+-\rangle] \Rightarrow B = \hbar^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

diagonalizing: eigenvectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} [|+-\rangle + |-+\rangle]$ w/ eigenvalue 2
 $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} [|+-\rangle - |-+\rangle]$ w/ "

$$ii) \text{ Use ladder operator } S_- |S=1, M=1\rangle = \sqrt{1(+1) - (1(-1))} \hbar |S=1, M=0\rangle$$

$$\rightarrow |S=1, M=0\rangle = \frac{1}{\sqrt{2}} [S_{1-} + S_{2-}] |++\rangle = \frac{1}{\sqrt{2}} \underbrace{\left[\frac{1}{2}\left(\frac{3}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right) \right]}_1 [|-+\rangle + |+-\rangle]$$

Last state $|S=0, M=0\rangle$ must be orthogonal to other 3 $\Rightarrow = \frac{1}{\sqrt{2}} [|+-\rangle - |-+\rangle]$

It can be checked directly that indeed $S^2 (|+-\rangle - |-+\rangle) = 0$

Summary

The state space generated by $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ (states with well-defined S_j^2, S_{jz}) is also generated by a basis of states with well defined total AM operators S^2, S_z

$$|S=1, M=1\rangle = |++\rangle$$

$$|S=1, M=0\rangle = \frac{1}{\sqrt{2}}[|+-\rangle - |-+\rangle] \quad \text{and} \quad |S=0, M=0\rangle = \underbrace{\frac{1}{\sqrt{2}}[|+-\rangle + |-+\rangle]}$$

$$|S=1, M=-1\rangle = |--\rangle$$

"triplet" states with $S=1$

"singlet" state w/ $S=0$

In other words, there are 2 different ways to combine 2 spin- $\frac{1}{2}$ particles one resulting in total AM corresponding to $S=1$, and the other to $S=0$

Note that triplet states are symmetric w/ respect to switching $1 \leftrightarrow 2$ whereas the singlet state is antisymmetric (gains a - sign)

Warning: the notation $|S=a, M=b\rangle$ does not imply any tensor product. Indeed note that states w/ $M=0$ are entangled w/ respect to local AM variables

IV - Summing arbitrary spins j_1 and j_2

Consider now 2 particles, one with spin j_1 and the other with spin j_2 . The state space is the $(2j_1+1)(2j_2+1)$ -dimensional space $\mathcal{E}_{j_1 j_2} = \mathcal{E}_{j_1} \otimes \mathcal{E}_{j_2}$ spanned by the products $|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |j_1, j_2, m_1, m_2\rangle$ (states with well-defined $J^2, J_{1z}, J_2^2, J_{2z}$)

The space $\mathcal{E}_{j_1 j_2}$ will also divide into a direct sum of $(2j+1)$ -dimensional subspaces, each spanned by states with well-defined J^2 and J_z

$$\mathcal{E}_{j_1 j_2} = \bigoplus_j \mathcal{E}_j$$

Our task is to find this redivision. To do this, there are 3 aspects to consider

i - We need to determine which values of j will appear in this sum

ii - There may be degeneracy in j , ie., for a given value of j we must check the possibility of there existing more than one space \mathcal{E}_j

iii - Each space \mathcal{E}_j is spanned by eigenstates $|j, m\rangle$ with well-defined J^2, J_z . We wish to find out how to write them in terms of the products $|j_1, j_2, m_1, m_2\rangle$

Note: since here j_1, j_2 are fixed, in the following we abbreviate where convenient:

$$|j_1, j_2, m_1, m_2\rangle \equiv |m_1, m_2\rangle$$

Before studying the problem mathematically, let us use our physical intuition

- we know that the largest possible value of $m = m_1 + m_2$ happens when each system is rotating as much as possible around \hat{z} , ie, when $m_1 = j_1$ and $m_2 = j_2 \rightarrow m = j_1 + j_2$
- we cannot have states with $j > j_1 + j_2$, otherwise there would have to exist a state $|j, m=j>j_1+j_2\rangle$. Therefore, the largest possible value of j is $j_1 + j_2$
- on the other hand, the smallest value of j (smallest total Am) must occur when each subsystem is rotating as much as possible around the same axis, but in opposite senses
 . If $j_1 = j_2$, this would mean a total $j=0$. However, since in general $j_1 \neq j_2$, there will always be at least $j = |j_1 - j_2|$ left.
- we can expect the values of j 's appearing in E_{jj_2} to range from $j = |j_1 - j_2|$ to $j = j_1 + j_2$

Note that $j_1 + j_2$ is $\begin{cases} \text{integer} \\ \gamma_2\text{-integer} \end{cases} \Leftrightarrow |j_1 - j_2|$ is $\begin{cases} \text{integer} \\ \gamma_2\text{-integer} \end{cases}$

∴ it is reasonable to expect that all values of $j \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2 - 1, j_1 + j_2\}$ should appear (there is no reason why not!)

To further support this intuition, we can simply count how many states $|j, m\rangle$ will exist if we allow one copy E_j for each j in this range. If these are indeed a complete set, their number should equal the dimension of the full space: ie: $\dim E_{j_1, j_2} = \dim(E_{j_1}) \times \dim(E_{j_2}) = (2j_1 + 1)(2j_2 + 1)$

Recall that each E_j contains $2j+1$ states $|j, m\rangle$.

If every $j \in \{j_1 - j_2, \dots, j_1 + j_2\}$ appears, the total number of basis states is:
(assuming $j_1 \geq j_2$ for simplicity)

$$\sum_{j=j_1-j_2}^{j_1+j_2} \dim(E_j) = \sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = 2 \left(\sum_{j=1}^{j_1+j_2} j - \sum_{j=1}^{j_1-j_2-1} j \right) + (j_1+j_2) - (j_1-j_2) + 1$$

$$\text{using } \sum_{j=1}^n j = \frac{n(n+1)}{2} : = \cancel{\left[\frac{(j_1+j_2)(j_1+j_2+1)}{2} - \frac{(j_1-j_2-1)(j_1-j_2)}{2} \right]} + 2j_2 + 1$$

$$= (j_1^2 + 2j_1j_2 + j_2^2 + j_1 + j_2) - (j_1^2 - 2j_1j_2 + j_2^2 - j_1 + j_2) + 2j_2 + 1$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1 = (2j_1 + 1)(2j_2 + 1) \stackrel{!}{=} (2k_1 + 1)(2k_2 + 1)$$

Let us now prove formally that this is indeed the correct solution, ie :

The space $E_{j_1 j_2}$ is spanned by a sum of exactly one copy of E_j for each value of $j \in \{ |j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2 \}$: $E_{j_1 j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} E_j$

(ie, actually there is no degeneracy in j within $E_{j_1 j_2}$)

Steps to prove this

- step 1 : check which values of $m = m_1 + m_2$ appear, and count their degeneracies

$$J_z |m_1, m_2\rangle = (J_{1z} + J_{2z}) |m_1, m_2\rangle = \underbrace{(m_1 + m_2)}_m \hbar |m_1, m_2\rangle$$

Since $m_1 \in \{-j_1, \dots, j_1\}$ and $m_2 \in \{-j_2, \dots, j_2\}$, then $\boxed{m \in \{-(j_1 + j_2), \dots, (j_1 + j_2)\}}$

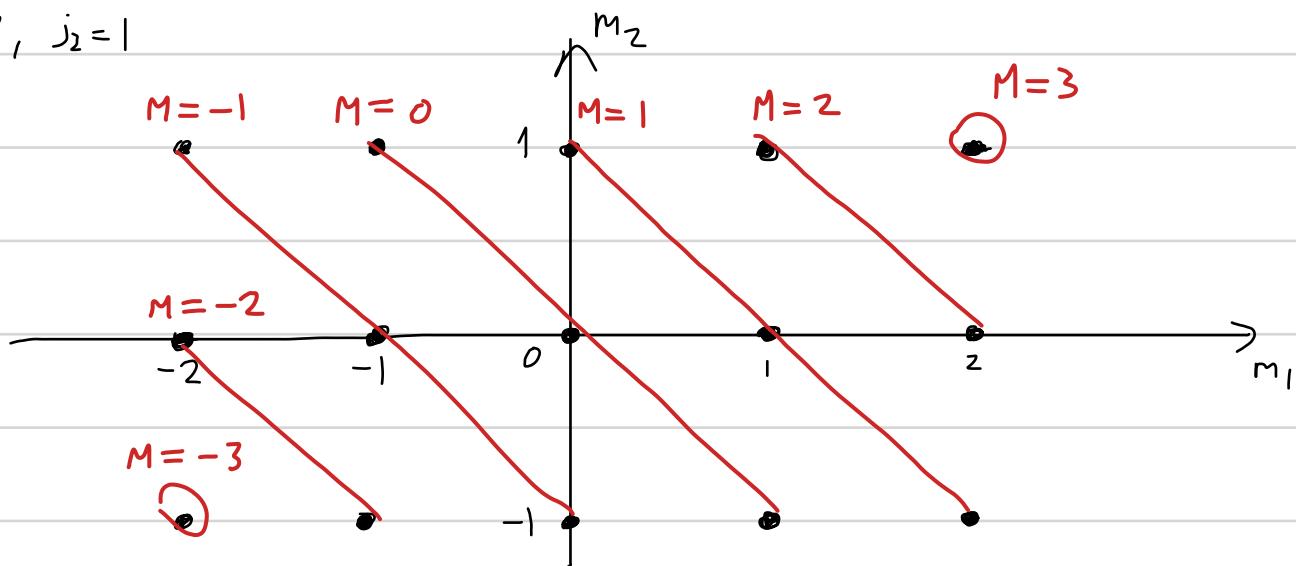
Furthermore, the degeneracy of a given value of m corresponds to the number of different ways to obtain m by combining m_1 and m_2 in these ranges

There is a useful geometric way to visualize this combinatorial problem:

Represent the possible values of m_1 on a horizontal axis, and those of m_2 on the vertical axis (note these values can be integer or half-integer)

For each value of m , the equation $m_1 + m_2 = m$ corresponds to a diagonal line connecting the points $(m_1, 0)$ and $(0, m_2)$. The degeneracies we want correspond to the number of points (m_1, m_2) with $-j_1 \leq m_1 \leq j_1$ and $-j_2 \leq m_2 \leq j_2$ that belong to each line.

ex: for $j_1 = 2, j_2 = 1$



- Analyzing more closely: the highest possible value for m (here $m=3$, in general $m=j_1+j_2$) can only be achieved in 1 way ($m_1=j_1, m_2=j_2$)
- As we move down to $m=2, m=1$, the degeneracy increases by 1
- It then stays constant ($=3=2j_2+1$) for $m=0$ and $m=-1$
- Finally, it decreases by 1 again in each of the next 2 steps (to $m=-2$ and $m=-3$) until, at $m=-3=-(j_1+j_2)$, there is again only 1 possibility ($m_1=-j_1, m_2=-j_2$)

Generalizing for arbitrary j_1, j_2 (we assume $j_1 \geq j_2$ for definiteness)

- Start from the highest value: $m=j_1+j_2 \Rightarrow$ only 1 way to obtain it [$m_1=j_1, m_2=j_2$]
- 2 ways to obtain next highest value ($m=j_1+j_2-1$): $m_1=j_1, m_2=j_2-1$ or $m_1=j_1-1, m_2=j_2$

- Continuing down, each time we lower the value of m by 1, there is one more way to obtain it, as long as j_2 is large enough

ex: to obtain $m = j_1 + j_2 - k$ ($k \in 0, 1, 2 \dots$) we can have:

$$\left\{ \begin{array}{l} m_1 = j_1 - k, m_2 = j_2 \\ m_1 = j_1 - k+1, m_2 = j_2 - 1 \\ \vdots \quad \vdots \\ m_1 = j_1, m_2 = j_2 - k \end{array} \right\} \begin{array}{l} k+1 \\ \text{different} \\ \text{combinations} \end{array}$$

This can only go on as long as $k \leq 2j_2$, so that $m_2 \geq -j_2$ (this limit is reached before the one for m_1 , since we are assuming $j_1 > j_2$). In particular, when $k = 2j_2$ ($\Leftrightarrow m = j_1 + j_2 - 2j_2 = j_1 - j_2$), there are $2j_2 + 1$ different ways of obtaining m

For $k > 2j_2$, we will still have $2j_2 + 1$ different possibilities, as long as j_1 is large enough

to obtain $m = j_1 + j_2 - k$:

$$\left\{ \begin{array}{l} m_1 = j_1 - k, m_2 = j_2 \\ \vdots \quad \vdots \\ m_1 = j_1 - k + 2j_2, m_2 = -j_2 \end{array} \right\} \begin{array}{l} 2j_2 + 1 \\ \text{different} \\ \text{values} \end{array}$$

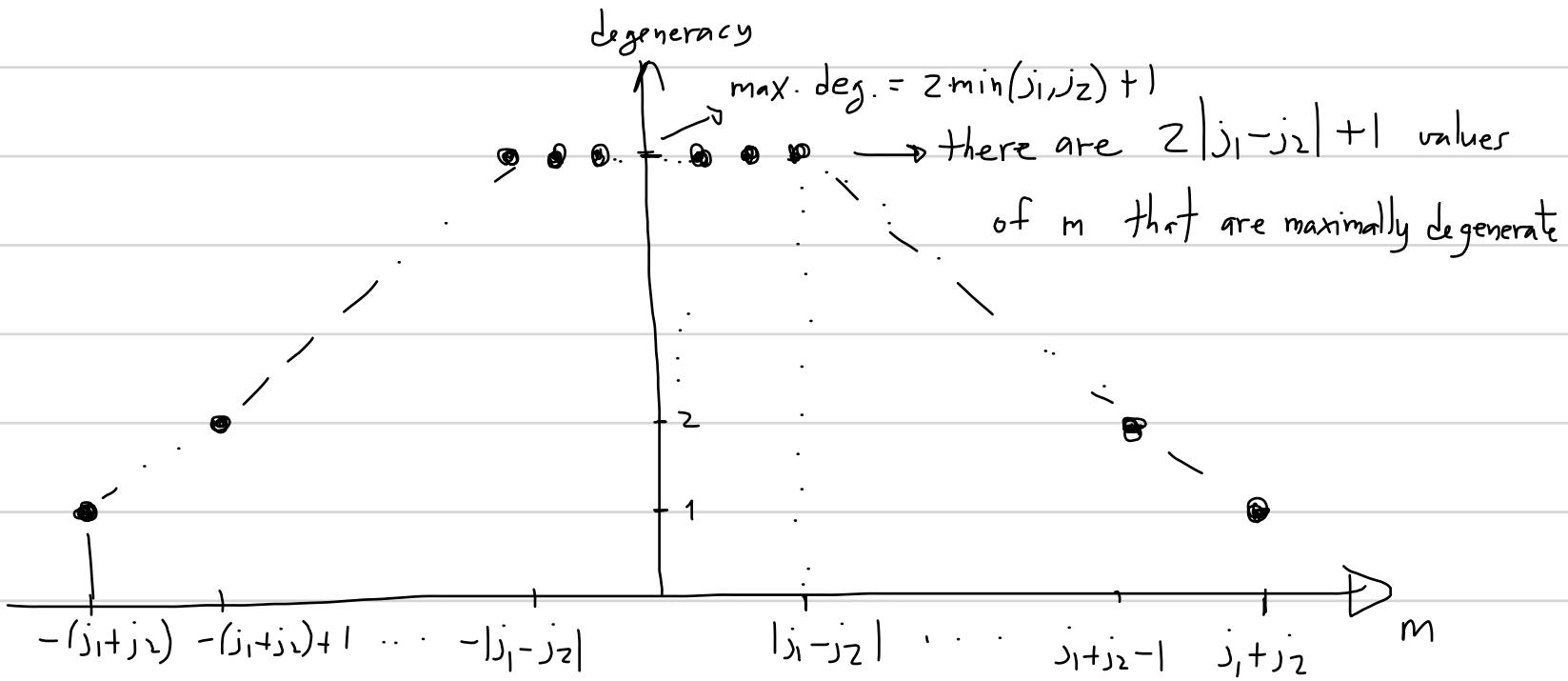
This can only go on as long as $m_1 \geq -j_1 \rightarrow k \leq 2j_1 \rightarrow m = j_2 - j_1 = -|j_1 - j_2|$

For $k > 2j_1$, each time we increase k by 1, we decrease the number of possibilities by 1. For example, for $k = 2j_1 + 1 \Leftrightarrow m = j_1 + j_2 - 2j_1 - 1 = -(j_1 - j_2) - 1$, we have

$$\left\{ \begin{array}{l} m_1 = -j_1, m_2 = j_2 - 1 \\ \vdots \quad \vdots \\ m_1 = -j_1 + 2j_2 - 1, m_2 = -j_2 \end{array} \right\} \begin{array}{l} 2j_2 \text{ possibilities} \end{array}$$

This goes on until $k = 2j_1 + 2j_2 = 2(j_1 + j_2) \Leftrightarrow m = -(j_1 + j_2)$, where there is again only 1 possibility ($m_1 = -j_1, m_2 = -j_2$)

In summary, the set of degeneracies can be represented as follows



- step 2: check which values of j can appear, and with which degeneracy
 - since we have no states with $|m| > j_1 + j_2$, then clearly we cannot have states with $j > j_1 + j_2$
 - since we have exactly one state with $m = j_1 + j_2$, we must have exactly one subspace E_j with $j = j_1 + j_2$ (which contains states $|j=j_1+j_2, m\rangle$ with $-(j_1+j_2) \leq m \leq (j_1+j_2)$)
 - since we still have exactly one more way to obtain a state with $m = j_1 + j_2 - 1$, then there must also exist exactly one subspace $E_{j_1+j_2-1}$
 - continuing the same argument we find there must be exactly one subspace E_j for each $j \in |j_1-j_2|, |j_1-j_2|+1, \dots, j_1+j_2-1, j_1+j_2$ and none for $j < |j_1-j_2|$ or $j > j_1+j_2$. Note steps always size 1 (j all integer or all half integer)

Example: in the case $j_1=2, j_2=1$ represented above, j ranges from

$$j = 2-1 = 1 \text{ to } j = 2+1 = 3 : E_{2,1} = E_1 \oplus E_2 \oplus E_3$$

We can now turn to the last aspect of the problem:

iii- Each space E_j is spanned by eigenstates $|j, m\rangle$ with well-defined J^2, J_z . We wish to find out how to write them in terms of the products $|j_1, j_2, m_1, m_2\rangle$

We work as in the $j_1=j_2=j/2$ example

- step 1: start from largest j, m : there is only one way to obtain

$$m = j_1 + j_2, \text{ so}$$

$$|j=j_1+j_2, m=m_1+m_2\rangle = |m_1=j_1, m_2=j_2\rangle$$

- step 2: apply ladder operator $J_- = J_{1-} + J_{2-}$ to repeatedly obtain $|j=j_1+j_2, m\rangle$

$$\text{ex: } |j=j_1+j_2, m=m_1+m_2-1\rangle = \underbrace{\frac{\hbar}{\sqrt{j_1(j_1+1)-j_1(j_1-1)}} |m_1=j_1-1, m_2=j_2\rangle + \frac{\hbar}{\sqrt{j_2(j_2+1)-j_2(j_2-1)}} |m_1=j_1, m_2=j_2-1\rangle}_{\hbar\sqrt{(j_1+j_2)(j_1+j_2+1)-(j_1+j_2)(j_1+j_2-1)}}$$

$$= \frac{\sqrt{2j_1}}{\sqrt{2(j_1+j_2)}} |m_1=j_1-1, m_2=j_2\rangle + \frac{\sqrt{2j_2}}{\sqrt{2(j_1+j_2)}} |m_1=j_1, m_2=j_2-1\rangle$$

$$|j=j_1+j_2, m=m_1+m_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}} |m_1=j_1-1, m_2=j_2\rangle + \sqrt{\frac{j_2}{j_1+j_2}} |m_1=j_1, m_2=j_2-1\rangle$$

Note the state is automatically normalised

Further states up to $|j=j_1+j_2, m=-(j_1+j_2)\rangle$ are obtained by iteration

- Step 3 : Move to the next-highest value of j : $j = j_1 + j_2 - 1$, again start w/
highest $m = j_1 + j_2 - 1$

To construct $|j=j_1+j_2-1, m=j_1+j_2-1\rangle$, note that this value of m is $2x$ degenerate in E_{j_1, j_2} (see diagram). Since we have already found one such state inside $E_{j_1+j_2}$, it follows that the state we are looking for here in $E_{j_1+j_2-1}$ must be the only state orthogonal to it. Thus :

$$|j=j_1+j_2-1, m=j_1+j_2-1\rangle = \sqrt{\frac{j_2}{j_1 + j_2}} |m_1=j_1-1, m_2=j\rangle - \sqrt{\frac{j_1}{j_1 + j_2}} |m_1=j_1, m_2=j_2-1\rangle$$

We then again construct the other states in this subspace by repeatedly applying J_-

- Further steps : iterate steps 1-3, each time reducing j by 1, until you reach $j=|j_1-j_2|$

Note : at the start of each E_j , construct $|j, m=j\rangle$ by imposing that it must be orthogonal to all other states with the same m that have already been found (there will always be just one left)

ex: $|j=j_1+j_2-2, m=j_1+j_2-2\rangle$ must be a combination of the form

$$\propto |m_1=j_1, m_2=j_2-2\rangle + \beta |m_1=j_1-1, m_2=j_2-2\rangle + \gamma |m_1=j_1-2, m_2=j_2\rangle$$

that is orthogonal to both $|j=j_1+j_2, m=j_1+j_2-2\rangle$ and $|j=j_1+j_2-1, m=j_1+j_2-2\rangle$ (both of which have already been found).

Summary: the procedure above gives a systematic method to construct all quantities of the form

$$\langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle \equiv C_{m_1, m_2, m}^{j_1, j_2, j}$$

These are known as Clebsch-Gordan coefficients, and their values can be found in standardized tables (obeying certain conventions, e.g. all coefficients real).

They also possess several beautiful symmetry properties, which are studied by a mathematical discipline known as group representation theory.

Griffiths

TABLE 4.8: Clebsch-Gordan coefficients. (A square root sign is understood for every entry; the minus sign, if present, goes *outside* the radical.)

$1/2 \times 1/2$	$\begin{array}{ c c }\hline 1 & \\ \hline +1 & \\ \hline +1/2 & +1/2 \\ \hline 1 & 0 \\ \hline -1/2 & +1/2 \\ \hline -1/2 & -1/2 \\ \hline -1/2 & -1/2 \\ \hline \end{array}$	j	$\begin{array}{ c c }\hline j_1 & j_2 \\ \hline M \\ \hline \end{array}$	$\begin{array}{ c c }\hline m_1 & m_2 \\ \hline + (C_{m_1, m_2, m}^{j_1, j_2, j})^2 \\ \hline - \\ \hline \end{array}$
$1 \times 1/2$	$\begin{array}{ c c }\hline 3/2 & \\ \hline +3/2 & \\ \hline +1 & +1/2 \\ \hline 1 & \\ \hline +1/2 & +1/2 \\ \hline 0 & -1/2 \\ \hline -1/2 & \\ \hline -1/2 & -1/2 \\ \hline 0 & -1/2 \\ \hline -1 & +1/2 \\ \hline 1/3 & 2/3 \\ \hline 2/3 & -1/3 \\ \hline -1/2 & -1/2 \\ \hline -1 & -1/2 \\ \hline 1 \\ \hline \end{array}$			
2×1	$\begin{array}{ c c }\hline 3 & \\ \hline +3 & \\ \hline +2 & +1 \\ \hline 1 & \\ \hline +2 & +2 \\ \hline +2 & 0 \\ \hline +1 & -1 \\ \hline 1/3 & 2/3 \\ \hline 2/3 & -1/3 \\ \hline -1 & -1/2 \\ \hline -1 & -1/2 \\ \hline 1 \\ \hline \end{array}$			
1×1	$\begin{array}{ c c }\hline 2 & \\ \hline +2 & \\ \hline 2 & 1 \\ \hline +1 & +1 \\ \hline 1 & 1 \\ \hline +1 & 0 \\ \hline 0 & +1 \\ \hline 1/2 & 1/2 \\ \hline 1/2 & -1/2 \\ \hline -1 & 1 \\ \hline -1 & -1 \\ \hline 1/6 & 1/2 \\ \hline 2/3 & 0 \\ \hline -1/6 & -1/2 \\ \hline 1/3 & \\ \hline 0 & -1 \\ \hline -1 & 0 \\ \hline 1/2 & -1/2 \\ \hline -1 & -1 \\ \hline 0 & -1 \\ \hline -1 & 0 \\ \hline 1/2 & 1/2 \\ \hline -1 & -1 \\ \hline 1 \\ \hline \end{array}$			
$2 \times 1/2$	$\begin{array}{ c c }\hline 5/2 & \\ \hline +5/2 & \\ \hline +2 & 1/2 \\ \hline 1 & \\ \hline +2 & -1/2 \\ \hline +1 & +1/2 \\ \hline 1/5 & 4/5 \\ \hline 4/5 & -1/5 \\ \hline +1/2 & +1/2 \\ \hline +1 & -1/2 \\ \hline 0 & +1/2 \\ \hline -1 & +1/2 \\ \hline 2/5 & 3/5 \\ \hline 3/5 & -2/5 \\ \hline -1/2 & -1/2 \\ \hline 0 & -1/2 \\ \hline -1 & +1/2 \\ \hline 2/5 & -3/5 \\ \hline -3/2 & -3/2 \\ \hline 5/2 & 3/2 \\ \hline -5/2 & -1/2 \\ \hline \end{array}$			
$3/2 \times 1/2$	$\begin{array}{ c c }\hline 2 & \\ \hline +2 & \\ \hline +3/2 & +1/2 \\ \hline 1 & \\ \hline +3/2 & -1/2 \\ \hline +1/2 & +1/2 \\ \hline 1/4 & 3/4 \\ \hline 3/4 & -1/4 \\ \hline +1 & +1 \\ \hline +2 & -1/2 \\ \hline -2 & +1/2 \\ \hline 1/2 & 1/2 \\ \hline 1/2 & -1/2 \\ \hline -1 & -1/2 \\ \hline -2 & -1/2 \\ \hline 4/5 & 1/5 \\ \hline 1/5 & -4/5 \\ \hline -5/2 & -5/2 \\ \hline 1 \\ \hline \end{array}$			
$3/2 \times 1$	$\begin{array}{ c c }\hline 5/2 & \\ \hline +5/2 & \\ \hline +3/2 & -1 \\ \hline 1 & \\ \hline +3/2 & 0 \\ \hline +1/2 & +1 \\ \hline 2/5 & 3/5 \\ \hline 3/5 & -2/5 \\ \hline +1/2 & +1/2 \\ \hline +1 & -1 \\ \hline 1/2 & 1/10 \\ \hline 1/2 & -8/15 \\ \hline -1/2 & +11 \\ \hline 3/10 & -8/15 \\ \hline 1/6 & \\ \hline -1/2 & -1/2 \\ \hline 1/2 & -1/2 \\ \hline 2 & 1 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline 1/10 & 2/5 \\ \hline 1/10 & -1/3 \\ \hline -1/2 & -1/2 \\ \hline 1/2 & -1/2 \\ \hline 2 & 1 \\ \hline 1/2 & 1/2 \\ \hline 1/2 & -1/2 \\ \hline -1 & -1 \\ \hline 1/2 & 1/10 \\ \hline 1/2 & -3/10 \\ \hline -1/2 & 0 \\ \hline 3/5 & -1/15 \\ \hline 1/10 & -2/5 \\ \hline 1/2 & 1/2 \\ \hline -3/2 & 3/2 \\ \hline 2 & 1 \\ \hline -3/2 & -3/2 \\ \hline 1 \\ \hline \end{array}$			

Note : there does in fact exist an exact formula for calculating the CG coefficients, given any set of compatible j_1, j_2, j, m_1, m_2, m . It is known as the "Racah formula", after its discoverer (G. Racah).

Unfortunately, it is quite complicated ! [See A Messiah, "Quantum Mechanics"]

It can however be easily programmed into a computer, thus allowing easy computation of the CG coefficients.

V - Example : summing spin 1 and spin $\frac{1}{2}$ (cf complement AXI eq. 36)

Possible values of j : from $1 - \frac{1}{2} = \frac{1}{2}$ to $1 + \frac{1}{2} = \frac{3}{2}$

State space $E_{j=\frac{3}{2}} = E_{1,\frac{1}{2}} \oplus E_{\frac{1}{2},\frac{1}{2}}$ note: dimension of $E_{j=\frac{3}{2}} = 3 \times 2 = 6$
of $E_{1,\frac{1}{2}} \oplus E_{\frac{1}{2},\frac{1}{2}} = 2+4=6$

1) $E_{j=\frac{3}{2}}$

Start with $|j=\frac{3}{2}, m=\frac{3}{2}\rangle = |m_1=1, m_2=\frac{1}{2}\rangle = |1\rangle|+\rangle$

$$J_- |j=\frac{3}{2}, m=\frac{3}{2}\rangle = \sqrt{\frac{3}{2} \left(\frac{1}{2}\right)} |j=\frac{3}{2}, m=\frac{1}{2}\rangle$$

$$\rightarrow |j=\frac{3}{2}, m=\frac{1}{2}\rangle = \frac{1}{\sqrt{3}} \left[\sqrt{1 \cdot 2 - 1 \cdot 0} |0\rangle|+\rangle + \sqrt{\frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot \frac{1}{2}} |1\rangle|-\rangle \right]$$

$$|j=\frac{3}{2}, m=\frac{1}{2}\rangle = \sqrt{\frac{2}{3}} |0+\rangle + \sqrt{\frac{1}{3}} |1-\rangle$$

$$|j=\frac{3}{2}, m=-\frac{1}{2}\rangle = \frac{1}{\sqrt{\frac{3}{2} \cdot \frac{5}{2} + \frac{1}{2} \cdot \frac{1}{2}}} |1-\rangle$$

$$= \frac{1}{2} \left[\sqrt{\frac{2}{3}} \sqrt{1 \cdot 2 - 0} |-1+\rangle + \sqrt{\frac{1}{3}} \sqrt{1 \cdot 2 - 1 \cdot 0} |0-\rangle + \sqrt{\frac{2}{3}} \sqrt{\frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot \frac{1}{2}} |0-\rangle + 0 \right]$$

$$|j=\frac{3}{2}, m=-\frac{1}{2}\rangle = \frac{1}{2} \left[\frac{2}{\sqrt{3}} |-1+\rangle + \sqrt{\frac{2}{3}} |0-\rangle \right] = \frac{1}{\sqrt{3}} |-1+\rangle + \sqrt{\frac{2}{3}} |0-\rangle$$

(note symmetry w $m=+\frac{1}{2}$ state)

Finally: $|j=\frac{3}{2}, m=-\frac{3}{2}\rangle = |-1-\rangle$ (could also of course obtain by applying J_-)

(11) $E_{1/2}$

$$\text{state } |j=\frac{1}{2}, m=\frac{1}{2}\rangle = \alpha |0+\rangle + \beta |1-\rangle$$

but must be orthogonal to $|j=\frac{3}{2}, m=\frac{1}{2}\rangle$

$$\rightarrow \boxed{|j=\frac{1}{2}, m=\frac{1}{2}\rangle = \sqrt{\frac{1}{3}} |0+\rangle - \sqrt{\frac{2}{3}} |1-\rangle}$$

$$\text{Finally: } |j=\frac{1}{2}, m=-\frac{1}{2}\rangle = \frac{1}{\hbar} J_- |j=\frac{1}{2}, m=\frac{1}{2}\rangle$$

$$= \sqrt{2-0} \sqrt{\frac{1}{3}} |-1+\rangle - \sqrt{2-0} \sqrt{\frac{2}{3}} |0-\rangle + \sqrt{\frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot \frac{1}{2}} \sqrt{\frac{1}{3}} |0-\rangle$$

$$\boxed{|j=\frac{1}{2}, m=-\frac{1}{2}\rangle = \sqrt{\frac{2}{3}} |-1-\rangle - \sqrt{\frac{1}{3}} |0-\rangle}$$

(Could also obtain by imposing orthogonal to $|j=\frac{3}{2}, m=-\frac{1}{2}\rangle$)

VI - General case : combining arbitrary J_1 and J_2

Finally, the most general case to consider is when $\vec{J} = \vec{J}_1 + \vec{J}_2$, where each ang mom \vec{J}_i may have eigenstates with different j 's, and even several sets of eigenstates with the same j , distinguished by some other parameters (specifically, by some other observables commuting w/ ang momentum). For example, in a H atom, there are multiple eigenstates with the same l, m , and which are distinguishable via the label " n " (corresp. to energy)

Suppose, then, that there exist observables A_1 in E_1 and B_2 in E_2 , such that : i) $[A_1, \vec{J}_1] = 0 = [B_2, \vec{J}_2]$ and also

ii) $\{A_1, J_1^2, J_{1z}\}$ and $\{B_2, J_2^2, J_{2z}\}$ form C.S.C.O.'s in E_1 and E_2 , respectively

so that we can uniquely identify a complete eigenbasis in each space by labels $|a, j_1, m_1\rangle$ and $|b, j_2, m_2\rangle$ (where a, b are the eigenvalues of A, B)

ex: if \vec{J}_1 is the orbital AM of an electron in a H atom, A_1 could be the electrostatic Hamiltonian $p_z^2/2m - \frac{k e^2}{r}$

Now: since $[J^2, J_1^2] = [J^2, J_2^2] = [J_z, J_1^2] = [J_z, J_2^2] = [A_1, \vec{J}] = [B_2, \vec{J}] = 0$

we can always choose the eigenstates of J^2, J_z to also be eigenstates of J_1^2, J_2^2, A_1 and B_2 (i.e, to have definite values of j_1, j_2, a_1 and b_2)

This means that we can in fact split the space $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ into a direct sum of subspaces uniquely identified by these labels

$$\mathcal{E} = \bigoplus_{a,b,j_1,j_2} \mathcal{E}_{j_1,j_2}^{a,b}$$

Within each of these subspaces, we can then construct states $|j, m\rangle$ in the same way we did above (with $j \in |j_1-j_2| \dots j_1+j_2$)

The full set of eigenstates of J^2, J_z can thus be obtained in the form

$$|a, b, j_1, j_2, j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} c_{m_1 m_2 m}^{j_1 j_2 j} |a, j_1, m_1\rangle \otimes |b, j_2, m_2\rangle$$

where $c_{m_1 m_2 m}^{j_1 j_2 j} = \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$ are Clebsch-Gordan coefficients

(note they are independent of the labels a, b)

Example : eigenstates of H with well-defined $\vec{J} = \vec{\Sigma} + \vec{\Sigma}'$

In this case the states of \mathcal{E}_1 are labeled $|n, l, m\rangle$, where n corresponds to the energy $E_n = -\frac{13,6}{n^2}$ eV, and for each $l \in (0, 1, 2, \dots)$ the value of n can vary from $l+1$ to ∞ (equivalently, for each $n \in (1, 2, \dots)$ the value of l can vary from 0 to $n-1$). Note $\{l \leftrightarrow j_1\}$ in our previous notation
 $\{m \leftrightarrow m_1\}$

Meanwhile, the states of \mathcal{E}_2 are just $|l\pm\rangle$ (ie, we only have $j_2 = \gamma_2$)

Thus, for each n, l , we can construct eigenstates of J^2, J_z of the form

$$\left| n, l=j, j_2=\frac{1}{2}, j, M \right\rangle = \sum_{m=-l}^l C_{m+M}^{l, \frac{1}{2}, j} \left| n, l, m \right\rangle |+\rangle + C_{m-M}^{l, \frac{1}{2}, j} \left| n, l, m \right\rangle |-\rangle$$

↑
here we use
capital M to
distinguish from m

Note here j can only be equal to $l \pm \frac{1}{2}$ (or only $\frac{1}{2}$ if $l=0$)

The Clebsch-Gordan coefficients $C_{m \pm M}^{l, \frac{1}{2}, j}$ can be obtained by studying the composition of each spin- l with spin- $\frac{1}{2}$. See complement AXL for details.

- $l=0$ is trivial. $m=0$ too (there is no sum in m), and $j=\frac{1}{2}$ only ($|\gamma_2 - 0| \leq j \leq \gamma_2 + 0$)

Also $M=m$, so we simply have

$$\left| n, l=0, j=\frac{1}{2}, M=\pm \right\rangle = \left| n, 0, 0 \right\rangle |\pm\rangle$$

- for $l=1$, we can use the results of example V above, just changing the notation slightly. Recall summing $l=1$ and $j_2=\frac{1}{2}$ results in states with $j=\frac{3}{2}$ or $j=\frac{1}{2}$:

Example V notation

$$\begin{aligned} \left| j=\frac{3}{2}, m=\frac{3}{2} \right\rangle &= \left| 1 \right\rangle |+\rangle \\ \left| j=\frac{3}{2}, m=\frac{1}{2} \right\rangle &= \sqrt{\frac{2}{3}} \left| 0+ \right\rangle + \sqrt{\frac{1}{3}} \left| 1- \right\rangle \\ \left| j=\frac{3}{2}, m=-\frac{1}{2} \right\rangle &= \sqrt{\frac{1}{3}} \left| -1+ \right\rangle + \sqrt{\frac{2}{3}} \left| 0- \right\rangle \\ \left| j=\frac{3}{2}, m=-\frac{3}{2} \right\rangle &= \left| -1- \right\rangle \\ \left| j=\frac{1}{2}, m=\frac{1}{2} \right\rangle &= \sqrt{\frac{1}{3}} \left| 0+ \right\rangle - \sqrt{\frac{2}{3}} \left| 1- \right\rangle \\ \left| j=\frac{1}{2}, m=-\frac{1}{2} \right\rangle &= \sqrt{\frac{2}{3}} \left| -1- \right\rangle - \sqrt{\frac{1}{3}} \left| 0- \right\rangle \end{aligned}$$

H atom notation

$$\begin{aligned} \left| n, l=1, j=\frac{3}{2}, M=\frac{3}{2} \right\rangle &= \left| n, 1, 1 \right\rangle |+\rangle \\ \left| n, l=1, j=\frac{3}{2}, M=\frac{1}{2} \right\rangle &= \sqrt{\frac{2}{3}} \left| n, 1, 0 \right\rangle |+\rangle + \sqrt{\frac{1}{3}} \left| n, 1, 1 \right\rangle |-\rangle \\ \left| n, l=1, j=\frac{1}{2}, M=-\frac{1}{2} \right\rangle &= \sqrt{\frac{1}{3}} \left| n, 1, -1 \right\rangle |+\rangle + \sqrt{\frac{2}{3}} \left| n, 1, 0 \right\rangle |-\rangle \\ \left| n, l=1, j=\frac{1}{2}, M=-\frac{3}{2} \right\rangle &= \left| n, 1, -1 \right\rangle |-\rangle \\ \left| n, l=1, j=\frac{1}{2}, M=\frac{1}{2} \right\rangle &= \sqrt{\frac{1}{3}} \left| n, 1, 0 \right\rangle |+\rangle - \sqrt{\frac{2}{3}} \left| n, 1, 1 \right\rangle |-\rangle \\ \left| n, l=1, j=\frac{1}{2}, M=-\frac{1}{2} \right\rangle &= \sqrt{\frac{2}{3}} \left| n, 1, -1 \right\rangle |-\rangle - \sqrt{\frac{1}{3}} \left| n, 1, 0 \right\rangle |-\rangle \end{aligned}$$