Instituto de Física
Universidade Federal Fluminense

Curso de Ótica Quântica 2020-1

Notas de aula - 6

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Electromagnetic modes as harmonic oscillators

Wave Equation in vacuum

Vector potential
Lorentz gauge

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

$$\mathbf{A}(\mathbf{r},t) = A(t)\,\mathbf{u}(\mathbf{r})$$

$$\nabla^2 \mathbf{u} + k^2 \mathbf{u} = 0$$

$$\frac{\partial^2 A}{\partial t^2} + \omega^2 A = 0 \qquad (\omega = ck)$$

No charges or currents

$$\rho(\mathbf{r}) = 0$$

$$\mathbf{J}(\mathbf{r}) = \mathbf{0}$$

Scalar potential:

$$\varphi(\mathbf{r}) = 0$$

Helmholtz

Harmonic oscillator

Helmholtz Equation: spatial mode structure

Helmholtz	Spatial modes vector space
$\nabla^2 u_j + k^2 u_j = 0$	$u_j(\mathbf{r}) \in \{\varphi_{lmn}(\mathbf{r})\}$
$\{e^{i\mathbf{k}\cdot\mathbf{r}}\} \quad (k_x, k_y, k_z)$	Plane
$\{B_l(k_\rho \rho) e^{i(k_z z + l\phi)}\} \quad (l, k_\rho, k_z)$	Cylindrical
$\{b_l(kr) Y_l^m(\theta, \phi)\} \quad (l, m, k)$	Spherical
$\{LG_{pl}\, e^{ik_z z}\} \quad (p, l, k_z)$	Paraxial: LG
$\{HG_{mn}\, e^{ik_z z}\} \quad (m, n, k_z)$	Paraxial: HG

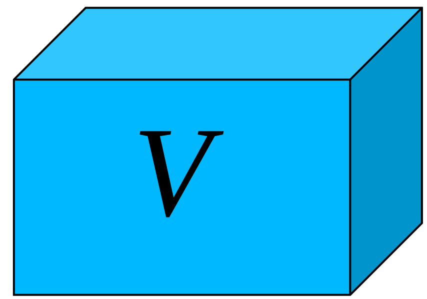
Further requirements

Physical fields

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Boundary conditions



Vector modes: Polarization-Spatial

$$\left\{\mathbf{u}_{jlmn}(\mathbf{r})\right\} \equiv \left\{\mathbf{u}_{\mu}(\mathbf{r})\right\}$$

Compact index

$$(j,l,m,n) \rightarrow \mu$$

$$\int_V \mathbf{u}_{\mu}^{*}(\mathbf{r}) \cdot \mathbf{u}_{\nu}(\mathbf{r})\, d^3\mathbf{r} = \delta_{\mu\nu}$$

Finite $V \longrightarrow$ discrete modes

Electromagnetic fields

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mu} A_{\mu}(t)\,\mathbf{u}_{\mu}(\mathbf{r})$$

$$\mathbf{E}(\mathbf{r},t) = -\sum_{\mu} \frac{dA_{\mu}}{dt}\,\mathbf{u}_{\mu}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r},t) = \sum_{\mu} A_{\mu}(t)\,\nabla \times \mathbf{u}_{\mu}$$

Hamiltonian

$$H = \int_{V} \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) d^3\mathbf{r}$$

Time evolution

Quadratures

$$\frac{\partial^2 A_\mu}{\partial t^2} + \omega_\mu^2 A_\mu = 0 \;\Rightarrow\; A_\mu(t) = \sqrt{\frac{\omega_\mu}{V\varepsilon_0}}\left(X_\mu \cos\omega_\mu t + Y_\mu \sin\omega_\mu t\right)$$

$$H = \int_V \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right) d^3\mathbf{r} = \sum_\mu H_\mu$$

$$H_\mu = \frac{\omega_\mu}{2}\left(X_\mu^2 + Y_\mu^2\right)$$

Harmonic Oscillator

Canonical quantization

Canonical quantization

Quadratures: canonical conjugate variables

$$\left.\begin{aligned} X_\mu &\to \hat{X}_\mu \\ Y_\mu &\to \hat{Y}_\mu \end{aligned}\right\} \Rightarrow \left[\hat{X}_\mu, \hat{Y}_\mu\right] = i\hbar$$

$$a_\mu = \frac{\hat{X}_\mu + i\hat{Y}_\mu}{\sqrt{2\hbar}}$$

Annihilation

$$a_\mu^\dagger = \frac{\hat{X}_\mu - i\hat{Y}_\mu}{\sqrt{2\hbar}}$$

Creation

$$N_\mu = a_\mu^\dagger a_\mu$$

Number

$$\left[a_\mu, a_\nu^\dagger\right] = \mathbf{1}\delta_{\mu\nu}$$

Fock states

$$N_\mu |n\rangle_\mu = n|n\rangle_\mu \qquad (n \in \mathbb{N})$$

$$a_\mu |n\rangle_\mu = \sqrt{n}\,|n-1\rangle_\mu$$

$$a_\mu^\dagger |n\rangle_\mu = \sqrt{n+1}\,|n+1\rangle_\mu$$

Quantum Hamiltonian

$$H_\mu = \hbar\omega_\mu \left(a_\mu^\dagger a_\mu + \frac{1}{2} \right)$$

Single mode

$$H_\mu \left| n \right\rangle_\mu = E_{n,\mu} \left| n \right\rangle_\mu$$

Eigenvectors: Fock states

$$E_{n,\mu} = \hbar\omega_\mu \left(n + \frac{1}{2} \right)$$

Eigenvalues

Heisenberg Equations

$$\frac{da_\mu}{dt} = \frac{1}{i\hbar}\left[a_\mu, H_\mu\right] = -i\omega_\mu a_\mu \quad \Rightarrow \quad \begin{cases} a_\mu(t) = a_\mu(0)\, e^{-i\omega_\mu t} \\ a_\mu^\dagger(t) = a_\mu^\dagger(0)\, e^{i\omega_\mu t} \end{cases}$$

$$\hat{X}_\mu(t) = \hat{X}_\mu(0)\cos\omega t + \hat{Y}_\mu(0)\sin\omega t$$
$$\hat{Y}_\mu(t) = -\hat{X}_\mu(0)\sin\omega t + \hat{Y}_\mu(0)\cos\omega t$$

Coherent states:

$$a_\mu|\alpha\rangle_\mu = \alpha|\alpha\rangle_\mu \quad \left(\alpha = |\alpha| e^{i\theta} \in \mathbb{C}\right)$$

"Classical-like"

$$x_\mu(t) = \langle\alpha|\hat{X}_\mu(t)|\alpha\rangle = |\alpha|\cos\left(\omega t - \theta\right)$$
$$y_\mu(t) = \langle\alpha|\hat{Y}_\mu(t)|\alpha\rangle = |\alpha|\sin\left(\omega t - \theta\right)$$

Minimum uncertainty

$$\langle\alpha|\left[\Delta\hat{X}_\mu(t)\right]^2|\alpha\rangle = \langle\alpha|\left[\Delta\hat{Y}_\mu(t)\right]^2|\alpha\rangle = \frac{\hbar}{2}$$

Quantum fluctuations

Photon number (energy)

$$\left\langle \left(\Delta N_\mu\right)^2 \right\rangle = \left\langle \psi \middle| \left(\Delta N_\mu\right)^2 \middle| \psi \right\rangle = \left\langle \psi \middle| N_\mu{}^2 \middle| \psi \right\rangle - \left\langle \psi \middle| N_\mu \middle| \psi \right\rangle^2$$

Fock states: $\left\langle \left(\Delta N_\mu\right)^2 \right\rangle = 0$

Coherent states: $\left\langle \left(\Delta N_\mu\right)^2 \right\rangle = |\alpha|^2$

Poisson distribution:

$$\left|\alpha\right\rangle_\mu = e^{-|\alpha|^2/2} \sum_m \frac{\alpha^m}{\sqrt{m!}} \left|m\right\rangle_\mu$$

$$P(n) = \left| \left\langle n \middle| \alpha \right\rangle_\mu \right|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

$$\left\langle N_\mu \right\rangle = |\alpha|^2 \qquad \sqrt{\left\langle \left(\Delta N_\mu\right)^2 \right\rangle} = |\alpha|^2 = \left\langle N_\mu \right\rangle$$

Quadrature

$$\left\langle \left(\Delta X_{\mu}\right)^{2}\right\rangle = \langle\psi|\left(\Delta X_{\mu}\right)^{2}|\psi\rangle = \langle\psi|X_{\mu}{}^{2}|\psi\rangle - \langle\psi|X_{\mu}|\psi\rangle^{2}$$

$$\left\langle \left(\Delta Y_{\mu}\right)^{2}\right\rangle = \langle\psi|\left(\Delta Y_{\mu}\right)^{2}|\psi\rangle = \langle\psi|Y_{\mu}{}^{2}|\psi\rangle - \langle\psi|Y_{\mu}|\psi\rangle^{2}$$

Fock states:

$$\langle X_{\mu}\rangle = \langle Y_{\mu}\rangle = 0$$

$$\left\langle \left(\Delta X_{\mu}\right)^{2}\right\rangle = \left\langle \left(\Delta Y_{\mu}\right)^{2}\right\rangle = \hbar\left(n+\frac{1}{2}\right)$$

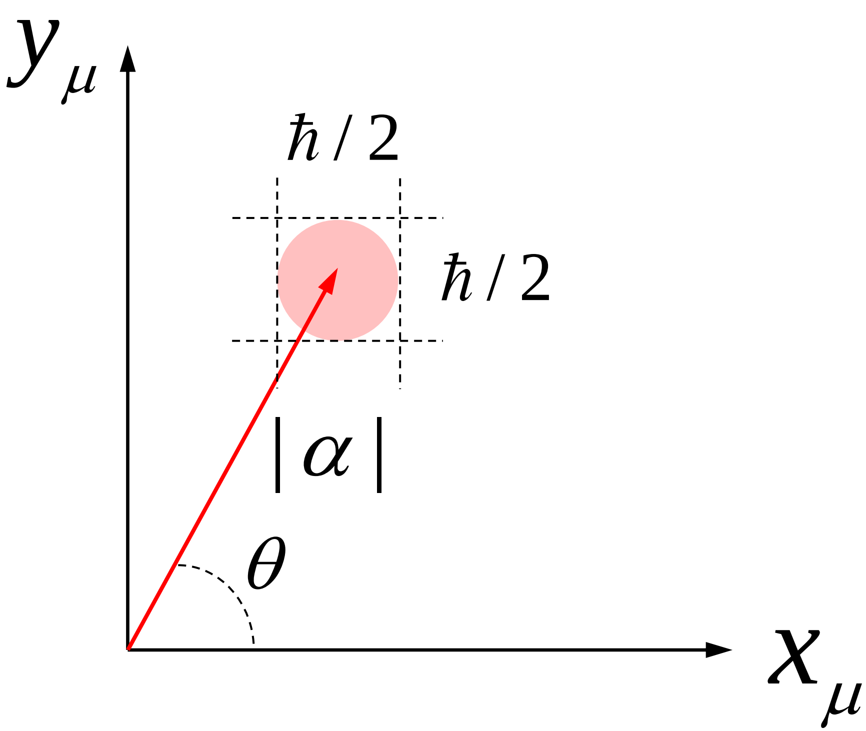
Coherent states:

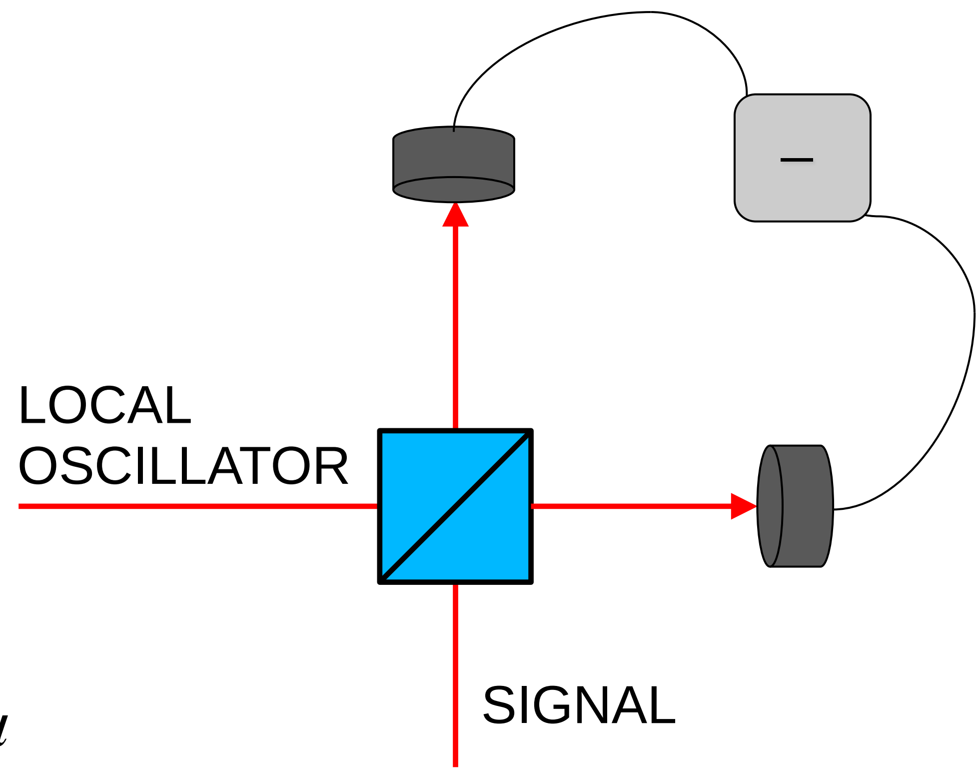
$$\langle X_{\mu}\rangle = \sqrt{\frac{\hbar}{2}}\left(\alpha+\alpha^{*}\right) \qquad \langle Y_{\mu}\rangle = -i\sqrt{\frac{\hbar}{2}}\left(\alpha-\alpha^{*}\right)$$

$$\left\langle \left(\Delta X_{\mu}\right)^{2}\right\rangle = \left\langle \left(\Delta Y_{\mu}\right)^{2}\right\rangle = \frac{\hbar}{2}$$

Phase space analysis

Quadrature measurement: homodyne detection

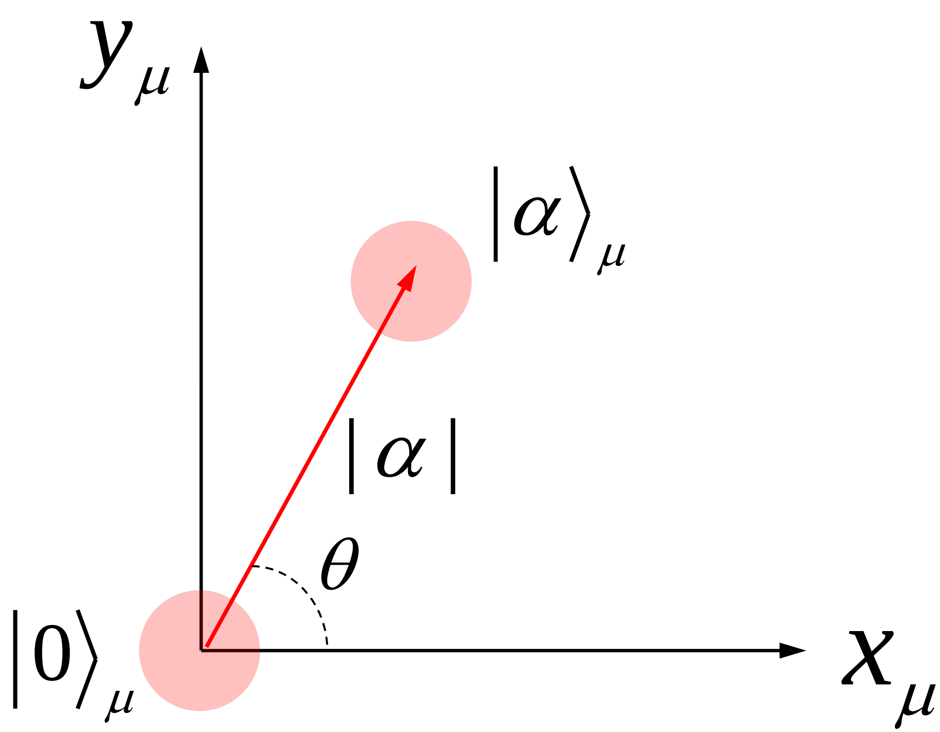




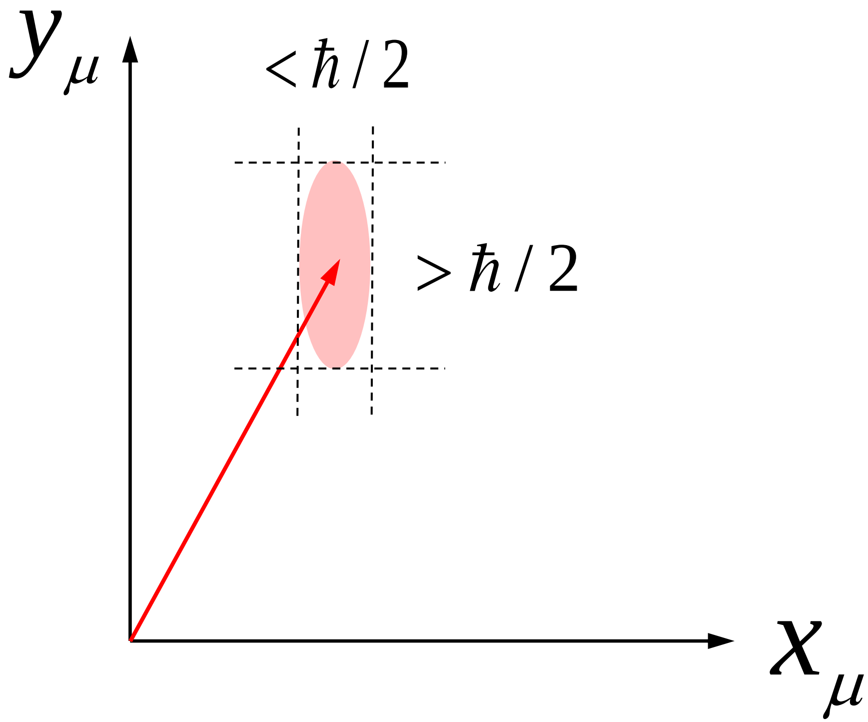
The Displacement Operator

$$D_\mu(\alpha) = \exp(\alpha^* a_\mu - \alpha a_\mu^\dagger) \quad \left(\alpha = |\alpha| e^{i\theta} \in \mathbb{C}\right)$$

$$D_\mu(\alpha)\left|0\right\rangle_\mu = \left|\alpha\right\rangle_\mu$$



Squeezed states



Minimum uncertainty

Squeezed states:

$$\sqrt{\left\langle \left(\Delta X_\mu\right)^2 \right\rangle \left\langle \left(\Delta Y_\mu\right)^2 \right\rangle} = \frac{\hbar}{2}$$

$$\left\langle \left(\Delta X_\mu\right)^2 \right\rangle \neq \left\langle \left(\Delta Y_\mu\right)^2 \right\rangle$$

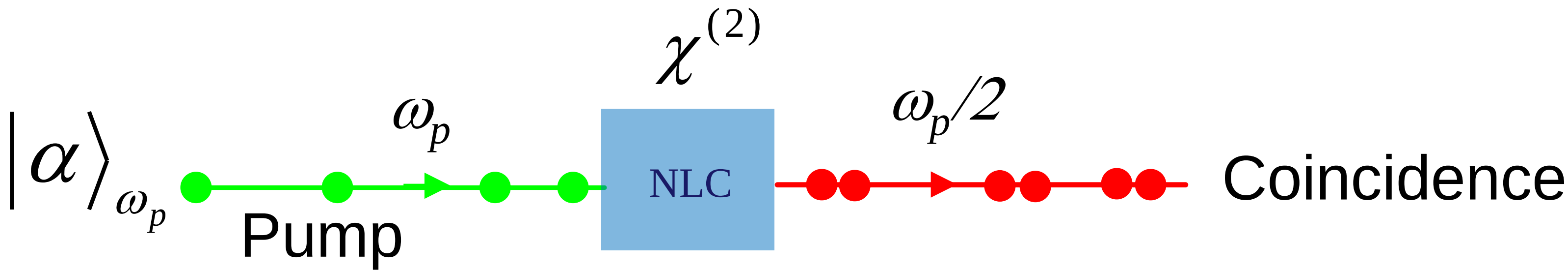
Squeeze operator

$$S_\mu(\xi) = \exp(\xi^* a_\mu^2 - \xi a_\mu^{2\dagger}) \quad \left(\xi = |\xi| e^{i\theta} \in \mathbb{C}\right)$$

$$S_\mu(\xi) |0\rangle_\mu = |\xi\rangle_\mu$$

Squeezed state generation

- Parametric down-conversion

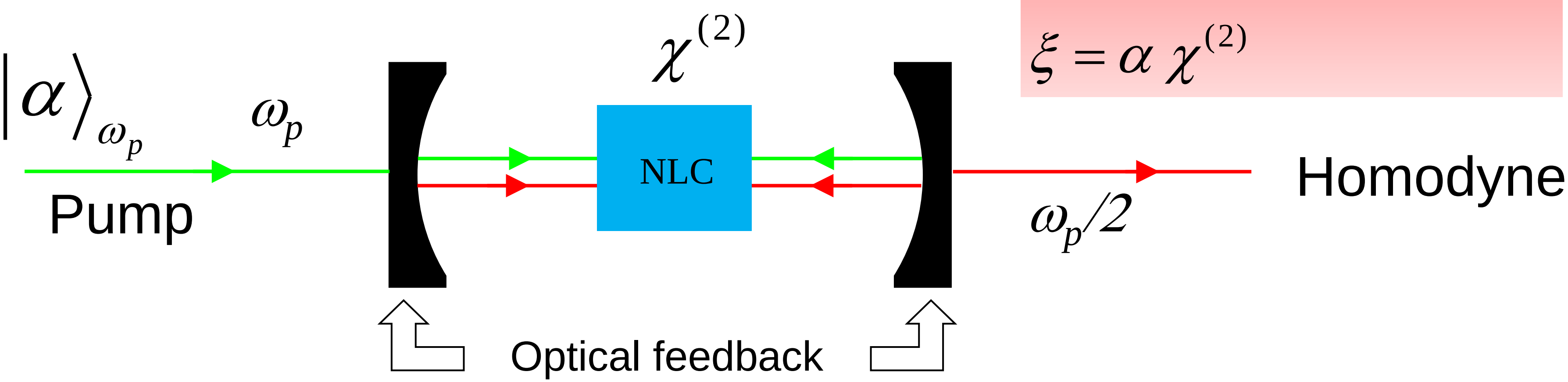


$$H_I = i\chi^{(2)}\left(a_{\omega_p} a^{2\dagger}_{\omega_p/2} - a^2_{\omega_p/2} a^\dagger_{\omega_p}\right) \approx i\chi^{(2)}\left(\alpha a^{2\dagger}_{\omega_p/2} - \alpha^* a^2_{\omega_p/2}\right)$$

- Optical parametric oscillator (OPO)

$$S_{\omega_p/2}(\xi)|0\rangle_{\omega_p/2} = |\xi\rangle_{\omega_p/2}$$

$$\xi = \alpha\,\chi^{(2)}$$



Multimode structure

Structure of the multimode vector space

Multimode vector space

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3 \cdots = \prod_{\mu}^{\otimes} \mathcal{E}_\mu$$

Multimode Fock states

$$|n_1, n_2, n_3, \ldots\rangle = |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes |n_3\rangle_3 \otimes \cdots \quad \Rightarrow \quad |\{n_\mu\}\rangle = \prod_{\mu}^{\otimes} |n_\mu\rangle_\mu$$

Operator extension

$$O_\nu \to O_\nu \otimes \prod_{\mu \neq \nu}^{\otimes} \mathbf{1}_\mu$$

$$a_\nu^\dagger a_\nu |\{n_\mu\}\rangle = n_\nu |\{n_\mu\}\rangle$$

Quantized fields

Electromagnetic fields

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{\mu} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\mu}}} \left[a_{\mu}(t)\mathbf{u}_{\mu}(\mathbf{r}) + a_{\mu}^{\dagger}(t)\mathbf{u}_{\mu}^{*}(\mathbf{r}) \right]$$

$$\hat{\mathbf{E}}(\mathbf{r},t) = i\sum_{\mu} \sqrt{\frac{\hbar\omega_{\mu}}{2\varepsilon_0 V}} \left[a_{\mu}(t)\mathbf{u}_{\mu}(\mathbf{r}) - a_{\mu}^{\dagger}(t)\mathbf{u}_{\mu}^{*}(\mathbf{r}) \right]$$

$$\hat{\mathbf{B}}(\mathbf{r},t) = \sum_{\mu} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\mu}}} \left[a_{\mu}(t)\nabla\times\mathbf{u}_{\mu} + a_{\mu}^{\dagger}(t)\nabla\times\mathbf{u}_{\mu}^{*} \right]$$

$$H = \sum_{\mu} \hbar\omega_{\mu} \left(a_{\mu}^{\dagger}a_{\mu} + \frac{1}{2} \right)$$

The vacuum state

$$|\text{vac}\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes \cdots = \prod_{\mu}^{\otimes} |0\rangle_\mu$$

Zero point energy

$$\langle \text{vac}| H |\text{vac}\rangle = \sum_\mu \frac{\hbar\omega_\mu}{2}$$

$$\langle \text{vac}| \hat{\mathbf{E}} |\text{vac}\rangle = 0$$

$$\langle \text{vac}| \Delta\hat{\mathbf{E}}^2 |\text{vac}\rangle \neq 0$$

⇒

Quantum fluctuations

Spontaneous emission

Casimir force

Multimode coherent states

Tensor product of coherent states

$$|\alpha_1\rangle_1 \otimes |\alpha_2\rangle_2 \otimes |\alpha_3\rangle_3 \otimes \cdots \equiv \prod_{\mu}^{\otimes} |\alpha_\mu\rangle_\mu \equiv |\alpha_1, \alpha_2, \alpha_3, \ldots\rangle \equiv |\{\alpha_\mu\}\rangle$$

$$a_\nu |\{\alpha_\mu\}\rangle = \alpha_\nu |\{\alpha_\mu\}\rangle$$

$$\langle\{\alpha_\mu\}|\hat{\mathbf{E}}(\mathbf{r},t)|\{\alpha_\mu\}\rangle = i\sum_\mu \sqrt{\frac{\hbar\omega_\mu}{2\varepsilon_0 V}} \left[\alpha_\mu(t)\mathbf{u}_\mu(\mathbf{r}) - \alpha_\mu^*(t)\mathbf{u}_\mu^*(\mathbf{r})\right]$$

$$D(\alpha_1,\ldots,\alpha_n) = \exp(\alpha_1^* a_1 - \alpha_1 a_1^\dagger + \ldots + \alpha_n^* a_n - \alpha_n a_n^\dagger) = D(\alpha_1)\cdots D(\alpha_n)$$

$$D(\alpha_1,\ldots,\alpha_n)|\text{vac}\rangle = |\alpha\rangle_1 \otimes \cdots \otimes |\alpha\rangle_n$$

Multimode displacement

Simple example

Example: plane wave, *x*-polarized, propagating along *z*

$$\mathbf{u}_{x,0,0,k}(\mathbf{r}) = e^{ikz}\,\hat{\mathbf{x}}$$

$$\alpha_{x,0,0,k} = \alpha$$

$$\alpha_\mu = 0 \quad \text{for} \quad \mu \neq (x,0,0,k)$$

$$\left\langle \{\alpha_\mu\} \middle| \hat{\mathbf{E}}(\mathbf{r},t) \middle| \{\alpha_\mu\} \right\rangle = \sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}}\,|\alpha(0)|\sin(kz-\omega t+\theta)\,\hat{\mathbf{x}}$$

Classical-like behaviour!

Mode transformations

Alternative modes

$$\{\mathbf{u}_\mu(\mathbf{r})\} \Leftrightarrow \{\mathbf{v}_\mu(\mathbf{r})\}$$

$$\int_V \mathbf{u}^*_\mu(\mathbf{r})\cdot\mathbf{u}_\nu(\mathbf{r})\, d^3\mathbf{r} = \delta_{\mu\nu}$$

$$\int_V \mathbf{v}^*_\mu(\mathbf{r})\cdot\mathbf{v}_\nu(\mathbf{r})\, d^3\mathbf{r} = \delta_{\mu\nu}$$

Mode transformation

$$\mathbf{v}_\mu(\mathbf{r}) = \sum_\nu U_{\mu\nu}\mathbf{u}_\nu(\mathbf{r})$$

$U \longrightarrow$ unitary

$$\mathbf{u}_\mu(\mathbf{r}) = \sum_\nu U^\dagger_{\mu\nu}\mathbf{v}_\nu(\mathbf{r})$$

OBS: $\omega_\mu = \omega_\nu$

Alternative modes

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_\mu \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_\mu}} a_\mu(t)\mathbf{u}_\mu(\mathbf{r}) + \text{h.c.}$$

$$= \sum_\mu \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_\mu}} a_\mu(t) \sum_\nu \left(U^\dagger\right)_{\mu\nu} \mathbf{v}_\nu(\mathbf{r}) + \text{h.c.}$$

$$= \sum_\nu \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_\nu}} \left[\sum_\mu \left(U^\dagger\right)_{\mu\nu} a_\mu(t)\right] \mathbf{v}_\nu(\mathbf{r}) + \text{h.c.}$$

$$= \sum_\nu \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_\nu}} \left[\sum_\mu U^*_{\nu\mu} a_\mu(t)\right] \mathbf{v}_\nu(\mathbf{r}) + \text{h.c.}$$

$$= \sum_\nu \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_\nu}} b_\nu(t)\, \mathbf{v}_\nu(\mathbf{r}) + \text{h.c.}$$

$$b_\nu(t) = \sum_\mu U^*_{\nu\mu} a_\mu(t) \quad \Rightarrow \quad \left[b^\dagger_\alpha, b_\beta\right] = \delta_{\alpha\beta}$$

Show this

Alternative modes

$$b_\nu = \sum_\mu U^*_{\nu\mu} a_\mu \Rightarrow b^\dagger_\nu = \sum_\mu U_{\nu\mu} a^\dagger_\mu$$

$$\sum_\nu b^\dagger_\nu b_\nu = \sum_\nu \left(\sum_\mu U_{\nu\mu}\, a^\dagger_\mu \right)\left(\sum_{\mu'} U^*_{\nu\mu'}\, a_{\mu'} \right)$$

$$= \sum_{\mu\mu'} \underbrace{\left(\sum_\nu U^\dagger_{\mu'\nu}\, U_{\nu\mu} \right)}_{U^\dagger U = I \to \delta_{\mu\mu'}} a^\dagger_\mu a_{\mu'} = \sum_\mu a^\dagger_\mu a_\mu$$

$U \rightarrow$ unitary

$$H = \sum_\mu \hbar\omega_\mu \left(a^\dagger_\mu a_\mu + \frac{1}{2} \right) = \sum_\mu \hbar\omega_\mu \left(b^\dagger_\mu b_\mu + \frac{1}{2} \right)$$

Fock states transformation

$$\{\mathbf{u}_\mu(\mathbf{r})\} \Leftrightarrow \{\mathbf{v}_\nu(\mathbf{r})\}$$

$$\left|\{n_\mu\}\right\rangle \Leftrightarrow \left|\{n_\nu\}\right\rangle$$

$$b_\nu = \sum_\mu U^*_{\nu\mu} a_\mu \Rightarrow b^\dagger_\nu = \sum_\mu U_{\nu\mu} a^\dagger_\mu$$

$$\left|\{0_\nu\}\right\rangle = \left|\{0_\mu\}\right\rangle = \left|\mathrm{vac}\right\rangle$$

$$\left|n_\nu, \{0_{\nu'\neq\nu}\}\right\rangle = \frac{\left(b^\dagger_\nu\right)^n}{\sqrt{n!}}\left|\mathrm{vac}\right\rangle = \frac{1}{\sqrt{n!}}\left(\sum_\mu U_{\nu\mu} a^\dagger_\mu\right)^n \left|\mathrm{vac}\right\rangle$$

$$\frac{1}{\sqrt{n!}}\left(\sum_\mu U_{\nu\mu} a^\dagger_\mu\right)^n \left|\mathrm{vac}\right\rangle = \frac{n!}{\sqrt{n!}} \sum_{\{m_\mu\}} \prod_\mu \frac{\left(U_{\nu\mu} a^\dagger_\mu\right)^{m_\mu}}{m_\mu !}\left|\mathrm{vac}\right\rangle = \sqrt{n!} \sum_{\{m_\mu\}} \prod_\mu^{\otimes} \frac{\left(U_{\nu\mu}\right)^{m_\mu}}{\sqrt{m_\mu !}}\left|m_\mu\right\rangle_\mu$$

where: $\sum_\mu m_\mu = n$

Coherent states transformation

$$\{\mathbf{u}_\mu(\mathbf{r})\} \Leftrightarrow \{\mathbf{v}_\nu(\mathbf{r})\}$$

$$\left|\{\alpha_\mu\}\right\rangle \Leftrightarrow \left|\{\beta_\nu\}\right\rangle$$

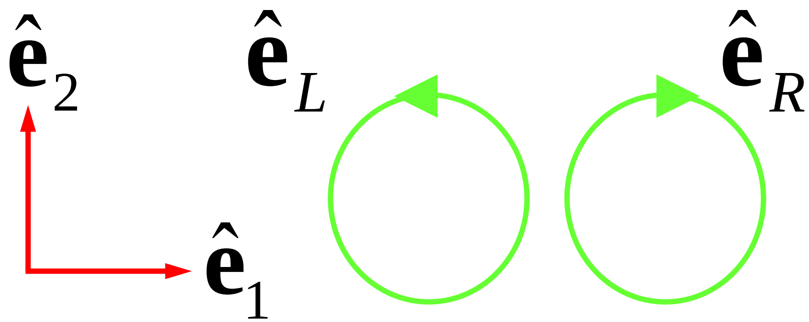
$$b_\nu = \sum_\mu U^*_{\nu\mu} a_\mu \Rightarrow b^\dagger_\nu = \sum_\mu U_{\nu\mu} a^\dagger_\mu$$

$$\left|\beta_\nu, \{0_{\nu'\neq\nu}\}\right\rangle = e^{\beta_\nu b^\dagger_\nu - \beta^*_\nu b_\nu} \left|\text{vac}\right\rangle = \underbrace{\exp\left(\sum_\mu \beta_\nu U_{\nu\mu} a^\dagger_\mu - \beta^*_\nu U^*_{\nu\mu} a_\mu\right)}_{\prod_\mu^{\otimes} D_\mu(\alpha_\mu)} \left|\text{vac}\right\rangle$$

$$\left|\beta_\nu, \{0_{\nu'\neq\nu}\}\right\rangle = \left|\{\alpha_\mu\}\right\rangle \text{ where } \alpha_\mu = \beta_\nu U_{\nu\mu}$$

Simple examples

Circular polarization modes



$$\hat{\mathbf{e}}_L = \frac{\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2}{\sqrt{2}}$$

$$\hat{\mathbf{e}}_R = \frac{\hat{\mathbf{e}}_1 - i\hat{\mathbf{e}}_2}{\sqrt{2}}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

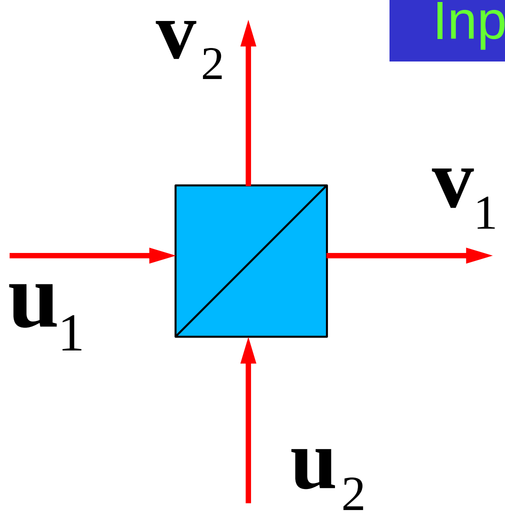
$$b_L = \frac{a_1 - ia_2}{\sqrt{2}} \qquad b_L^\dagger = \frac{a_1^\dagger + ia_2^\dagger}{\sqrt{2}}$$

$$b_R = \frac{a_1 + ia_2}{\sqrt{2}} \qquad b_R^\dagger = \frac{a_1^\dagger - ia_2^\dagger}{\sqrt{2}}$$

$$\Rightarrow \quad b_R\hat{\mathbf{e}}_R + b_L\hat{\mathbf{e}}_L = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2$$

Simple examples

Input-output modes of a beam splitter



$$\mathbf{v}_1 = \frac{\mathbf{u}_1 + \mathbf{u}_2}{\sqrt{2}}$$

$$\mathbf{v}_2 = \frac{-\mathbf{u}_1 + \mathbf{u}_2}{\sqrt{2}}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$b_1 = \frac{a_1 + a_2}{\sqrt{2}} \qquad b_1^\dagger = \frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}}$$

$$b_2 = \frac{a_1 - a_2}{\sqrt{2}} \qquad b_2^\dagger = \frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}$$

$$\Rightarrow \quad b_1\mathbf{v}_1 + b_2\mathbf{v}_2 = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$$

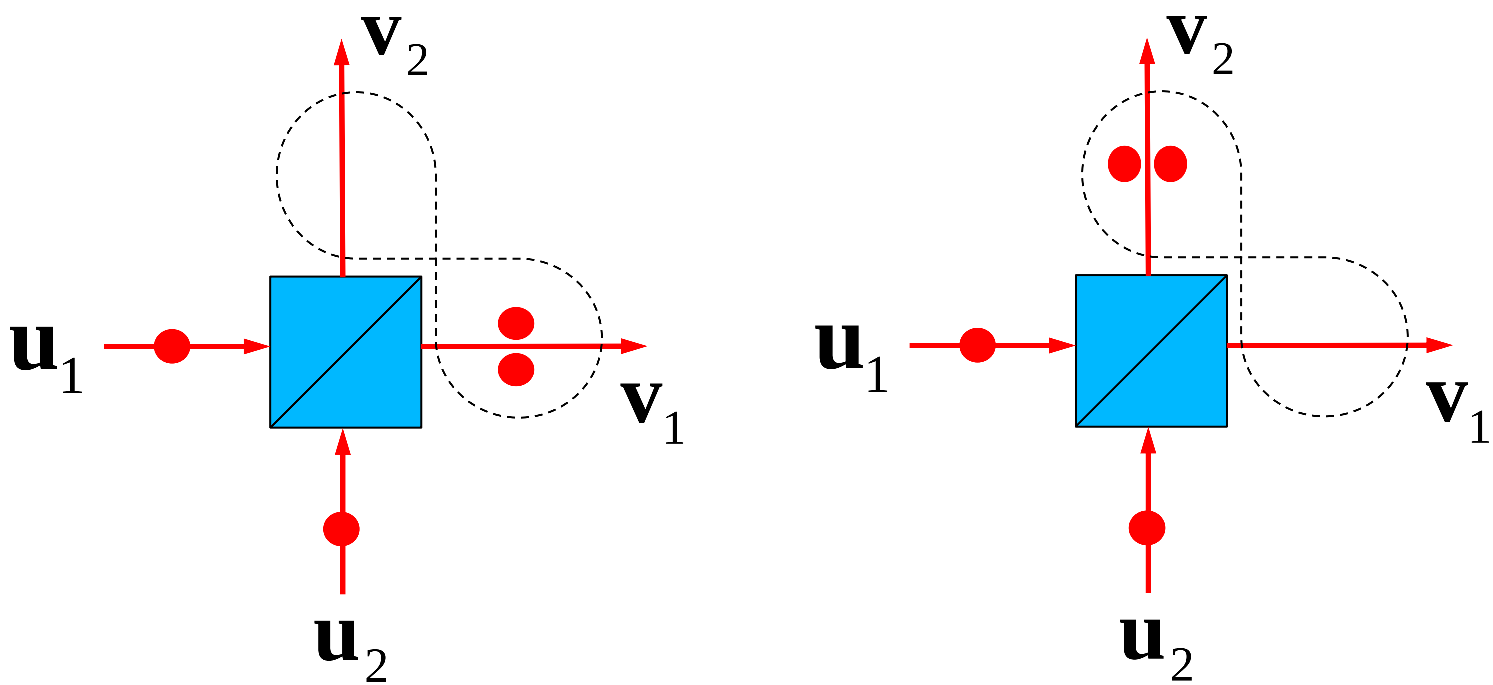
$$|n\rangle_{\mathbf{v}_1}|N-n\rangle_{\mathbf{v}_2} = \frac{\left(b_{\mathbf{v}_1}^\dagger\right)^n \left(b_{\mathbf{v}_2}^\dagger\right)^{N-n}}{\sqrt{n!(N-n)!}}|\mathrm{vac}\rangle = \frac{1}{\sqrt{n!(N-n)!}}\left(\frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}}\right)^n \left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}\right)^{N-n}|\mathrm{vac}\rangle$$

Hong-Ou-Mandel Effect

Input-output of a photon pair on a beam splitter

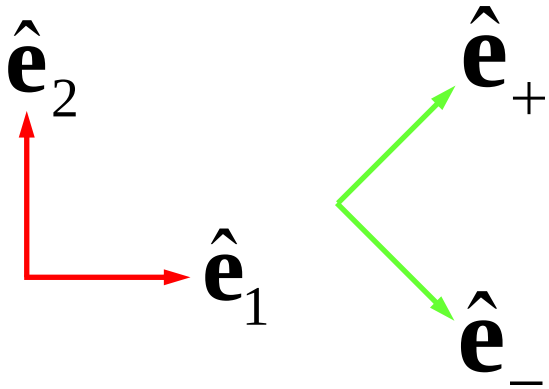
$$|1\rangle_{\mathbf{u}_1}|1\rangle_{\mathbf{u}_2} = a^\dagger_{\mathbf{u}_1} a^\dagger_{\mathbf{u}_2}|\text{vac}\rangle = \left(\frac{b^\dagger_{\mathbf{v}_1} + b^\dagger_{\mathbf{v}_2}}{\sqrt{2}}\right)\left(\frac{b^\dagger_{\mathbf{v}_1} - b^\dagger_{\mathbf{v}_2}}{\sqrt{2}}\right)|\text{vac}\rangle = \left(\frac{\left(b^\dagger_{\mathbf{v}_1}\right)^2 - \left(b^\dagger_{\mathbf{v}_2}\right)^2}{2}\right)|\text{vac}\rangle$$

$$|1\rangle_{\mathbf{u}_1}|1\rangle_{\mathbf{u}_2} = \frac{|2\rangle_{\mathbf{v}_1}|0\rangle_{\mathbf{v}_2} - |0\rangle_{\mathbf{v}_1}|2\rangle_{\mathbf{v}_2}}{\sqrt{2}}$$



Simple examples

Linear polarization modes



$$\hat{\mathbf{e}}_+ = \frac{\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{2}}$$

$$\hat{\mathbf{e}}_- = \frac{\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2}{\sqrt{2}}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$b_+ = \frac{a_1 + a_2}{\sqrt{2}} \qquad b_L^\dagger = \frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}}$$

$$b_- = \frac{a_1 - a_2}{\sqrt{2}} \qquad b_R^\dagger = \frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}$$

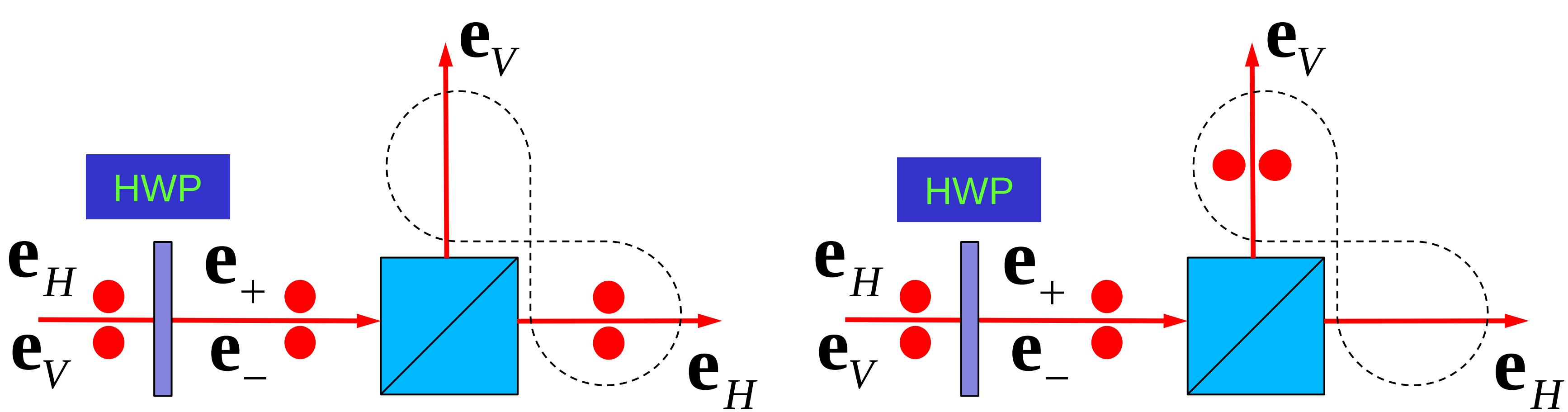
$$\Rightarrow \quad b_+\hat{\mathbf{e}}_+ + b_-\hat{\mathbf{e}}_- = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2$$

Polarization Hong-Ou-Mandel Effect

Input-output of a photon pair on a beam splitter

$$|1\rangle_{\mathbf{e}_+}|1\rangle_{\mathbf{e}_-} = a^\dagger_{\mathbf{e}_+} a^\dagger_{\mathbf{e}_-}|\text{vac}\rangle = \left(\frac{b^\dagger_{\mathbf{e}_H} + b^\dagger_{\mathbf{e}_V}}{\sqrt{2}}\right)\left(\frac{b^\dagger_{\mathbf{e}_H} - b^\dagger_{\mathbf{e}_V}}{\sqrt{2}}\right)|\text{vac}\rangle = \left(\frac{\left(b^\dagger_{\mathbf{e}_H}\right)^2 - \left(b^\dagger_{\mathbf{e}_V}\right)^2}{2}\right)|\text{vac}\rangle$$

$$|1\rangle_{\mathbf{e}_+}|1\rangle_{\mathbf{e}_-} = \frac{|2\rangle_{\mathbf{e}_H}|0\rangle_{\mathbf{e}_V} - |0\rangle_{\mathbf{e}_H}|2\rangle_{\mathbf{e}_V}}{\sqrt{2}}$$



Two-mode squeezed states

$$S_{\mu\nu}(\xi) = \exp(\xi^* a_\mu b_\nu - \xi a_\mu^\dagger b_\nu^\dagger) \quad \left(\xi = |\xi| e^{i\theta} \in \mathbb{C}\right)$$

$$S_{\mu\nu}(\xi) \left|0\right\rangle_\mu \left|0\right\rangle_\nu = \left|\xi\right\rangle_{\mu\nu}$$

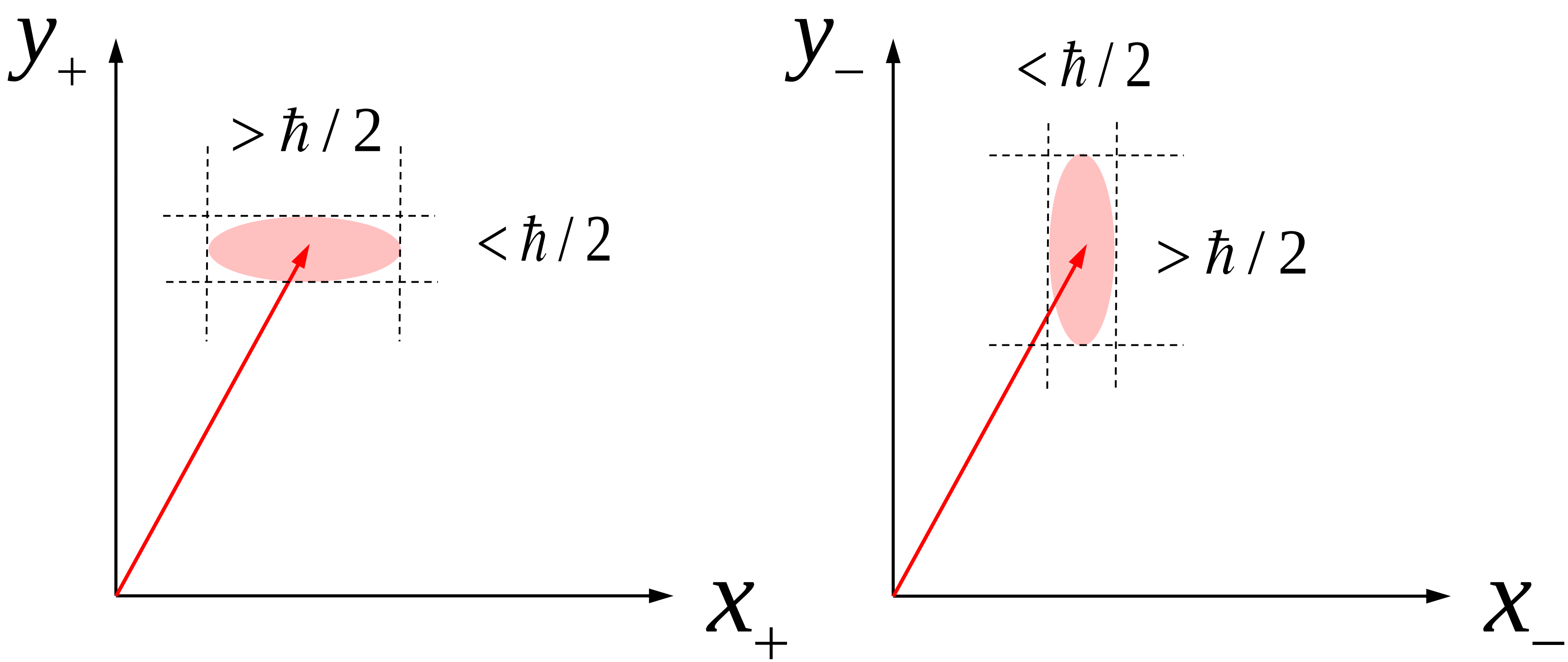
Quadrature entanglement

EPR variables:

$$X_\pm = \frac{X_\mu \pm X_\nu}{2} \quad Y_\pm = \frac{Y_\mu \pm Y_\nu}{2}$$

$$\left\langle (\Delta X_-)^2 \right\rangle + \left\langle (\Delta Y_+)^2 \right\rangle < \hbar \quad \text{(quadrature entanglement)}$$

Two-mode squeezed states

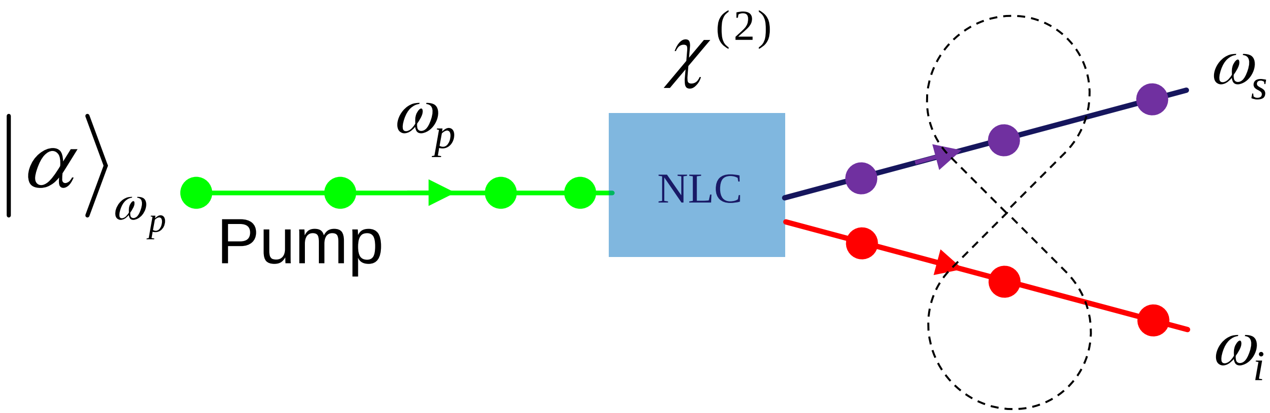


EPR correlations (quadrature entanglement):

$$\left\langle \left(\Delta X_-\right)^2 \right\rangle + \left\langle \left(\Delta Y_+\right)^2 \right\rangle < \hbar$$

Two-mode squeezed state generation

- Nondegenerate parametric down-conversion



$$H_I = i\chi^{(2)}\left(a_{\omega_p} b^\dagger_{\omega_s} b^\dagger_{\omega_i} - a^\dagger_{\omega_p} b_{\omega_s} b_{\omega_i}\right) \approx i\chi^{(2)}\left(\alpha b^\dagger_{\omega_s} b^\dagger_{\omega_i} - \alpha^* b_{\omega_s} b_{\omega_i}\right)$$

$$S_{\omega_s\omega_i}(\xi)|0\rangle_{\omega_p} = |\xi\rangle_{\omega_s\omega_i}$$

$$\xi = \chi^{(2)}\alpha$$