Instituto de Física
Universidade Federal Fluminense

Curso de Ótica Quântica 2020-1
Notas de aula - 5

Antonio Zelaquett Khoury

Paraxial Equation

(Paraxial Equation)

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\vec{E} = u(\vec{r})\,\hat{n}\,e^{-i\omega t}$$

$$\nabla^2 u + k^2 u = 0$$

$$u(\vec{r}) = \psi(x, y, z)\,e^{ikz}$$

$$\frac{\partial^2 \psi}{\partial z^2} << k\frac{\partial \psi}{\partial z}$$

$$\nabla_\perp^2 \psi + 2ik\frac{\partial \psi}{\partial z} = 0$$

Fundamental Gaussian Mode

$$\psi_0(\mathbf{r}) = \sqrt{\frac{2}{\pi}}\frac{1}{w(z)}\exp\left(-\frac{x^2+y^2}{w^2(z)}\right)\exp\left(ik\frac{x^2+y^2}{2R(z)}\right)e^{-i\arctan(z/z_R)}$$

Beam width

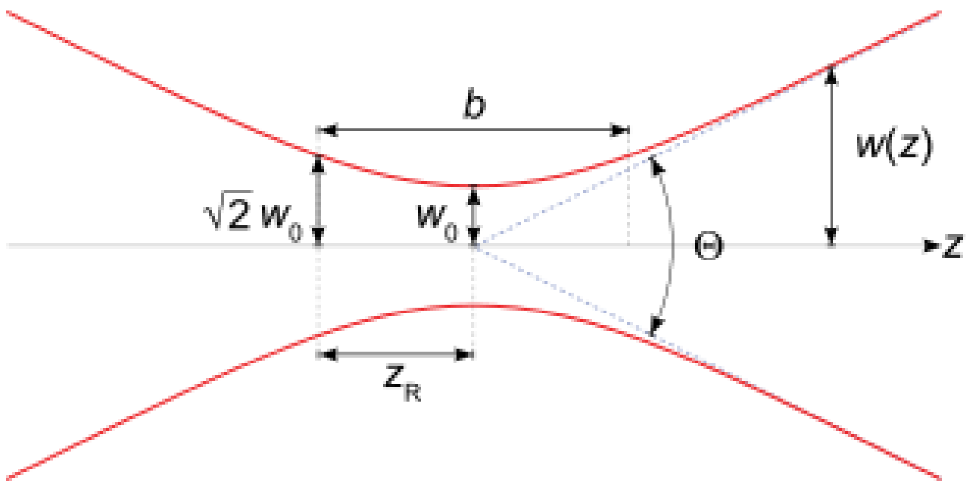
$$w(z) = w_0\sqrt{\left(1+\frac{z^2}{z_R^2}\right)}$$

Wavefront radius

$$R(z) = z\left(1+\frac{z_R^2}{z^2}\right)$$

Rayleigh range

$$z_R = \frac{\pi w_0}{\lambda}$$

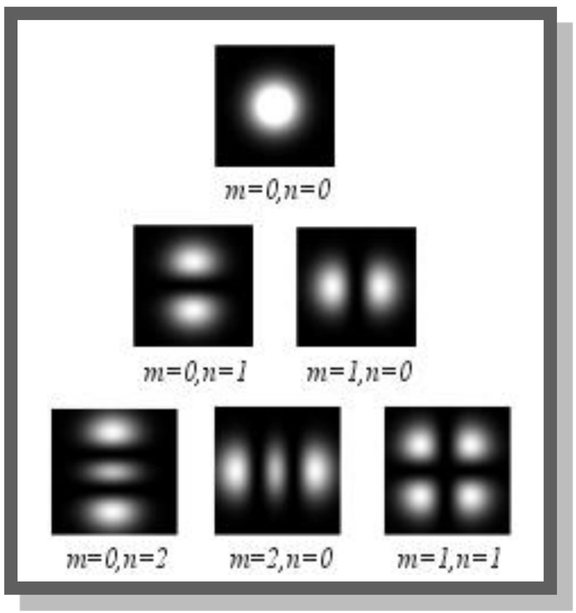


General Paraxial Modes

(Paraxial Equation)

$$\nabla_{\perp}^{2}\psi + 2ik\frac{\partial \psi}{\partial z} = 0$$

Hermite-Gauss (HG)

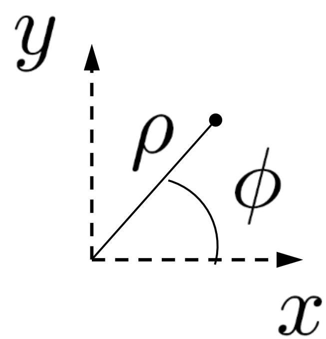


Rectangular

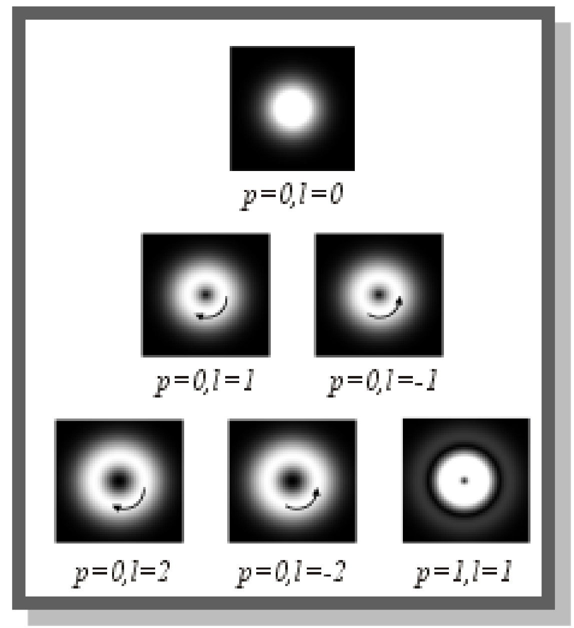
$HG_{nm}(x,y)$

Cylindrical

$LG_{pl}(\rho,\phi)$



Laguerre-Gauss (LG)



Hermite-Gaussian Modes

$$HG_{mn}(\mathbf{r}) = \frac{A_{mn}}{w(z)} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left(-\frac{x^2+y^2}{w^2(z)}\right) \exp\left(ik\frac{x^2+y^2}{2R(z)}\right) e^{-i\varphi_N(z)}$$

Gouy phase

$$\varphi_N(z) = (N+1)\arctan(z/z_R)$$

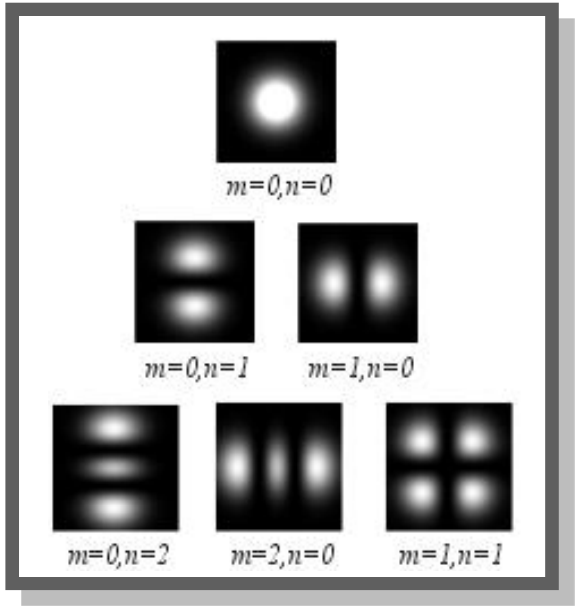
$N=m+n$

Orthonormal

$$\int HG^*_{mn}(\mathbf{r})\, HG_{m'n'}(\mathbf{r})\, d^2\mathbf{r} = \delta_{mm'}\,\delta_{nn'}$$

Complete

$$\sum_{m,n} HG^*_{mn}(\mathbf{r})\, HG_{mn}(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$



$N=0$

$N=1$

$N=2$

Laguerre-Gaussian Modes

$$LG_{pl}(\mathbf{r}) = \frac{A_{pl}}{w(z)}\left(\frac{\sqrt{2}\rho}{w(z)}\right)^{|l|} L_p^{|l|}\left(\frac{2\rho^2}{w^2(z)}\right)\exp\left(-\frac{\rho^2}{w^2(z)}\right)\exp\left(ik\frac{\rho^2}{2R(z)}\right)e^{-i\varphi_N(z)}e^{il\phi}$$

Gouy phase

$$\varphi_N(z) = (N+1)\arctan\left(z/z_R\right)$$

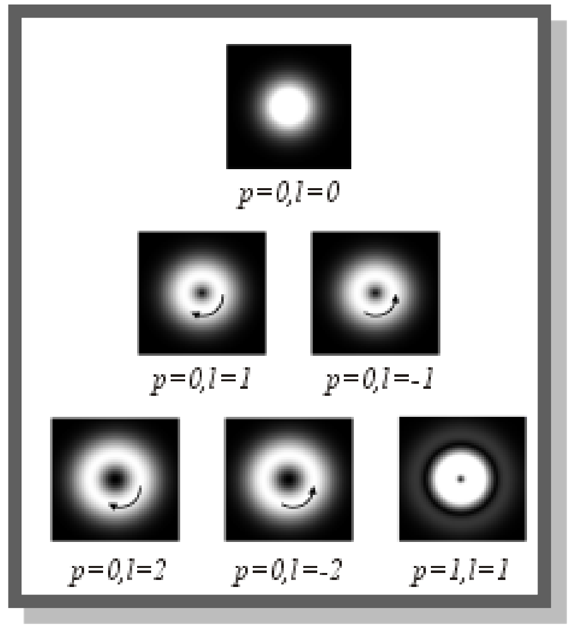
$N=2p+|l|$

Orthonormal

$$\int LG_{pl}^{*}(\mathbf{r})\, LG_{p'l'}(\mathbf{r})\, d^2\mathbf{r} = \delta_{pp'}\,\delta_{ll'}$$

Complete

$$\sum_{p,l} LG_{pl}^{*}(\mathbf{r})\, LG_{pl}(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$



$N=0$

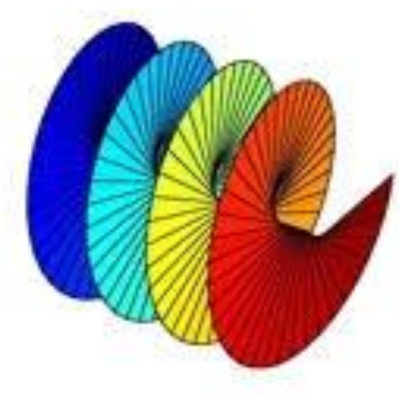
$N=1$

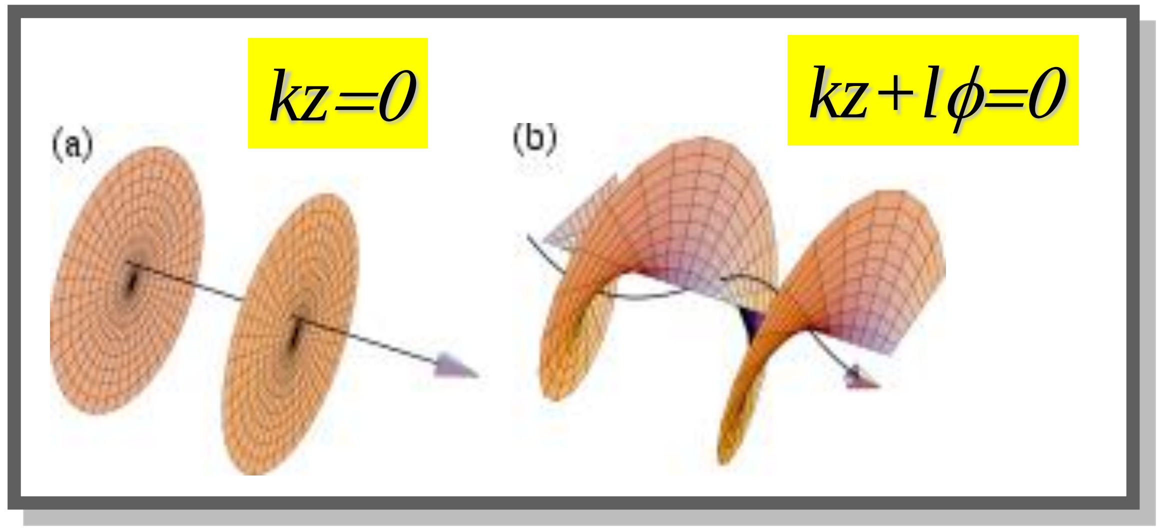
$N=2$

Orbital angular momentum

Twisted wavefront

l=1





$$\mathbf{P} = \varepsilon_0 \int \mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\, d^3\mathbf{r}$$

Linear momentum

$$\mathbf{J}(\mathbf{r}_0) = \varepsilon_0 \int (\mathbf{r} - \mathbf{r}_0) \times [\mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})]\, d^3\mathbf{r}$$

Angular momentum

Paraxial propagation

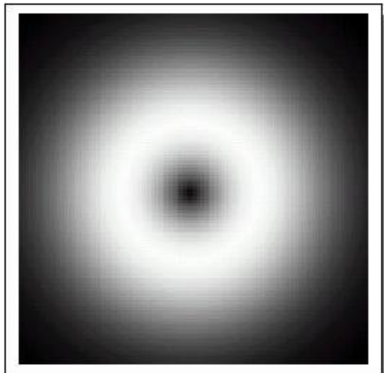
$$\mathbf{J} = \mathbf{J}_S + \mathbf{J}_O$$

SPIN + ORBITAL

pol *wavefront*

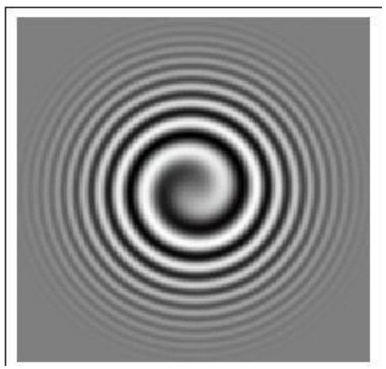
Intensity and phase of LG modes

Intensity



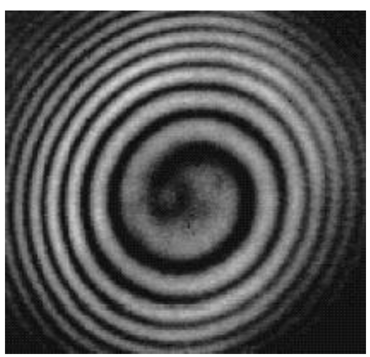
$$\left|LG_{0,\pm1}(r,\phi)\right|^2 \propto r^2 e^{-2r^2/w^2}$$

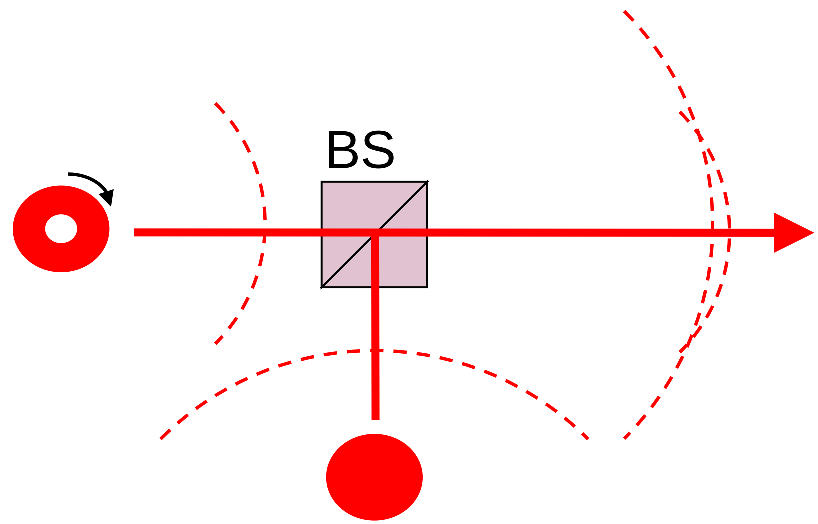
Phase (theo)



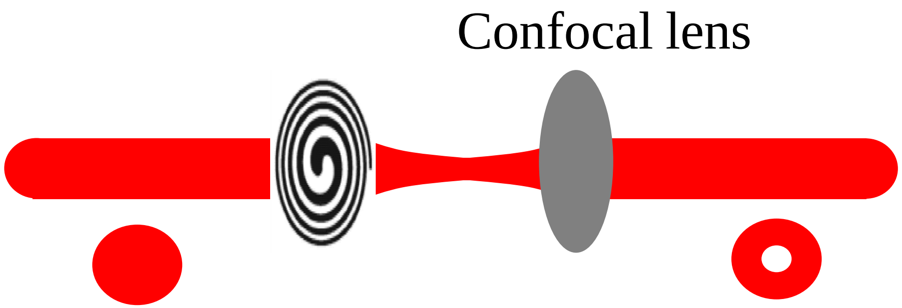
$$LG_{0,\pm1}(r,\phi) \propto r\, e^{\pm i\phi} e^{-r^2/w^2}$$

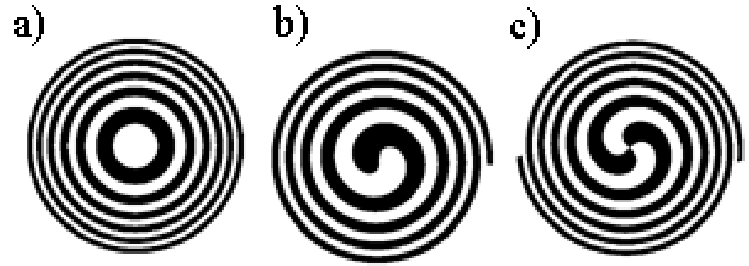
Phase (exp)



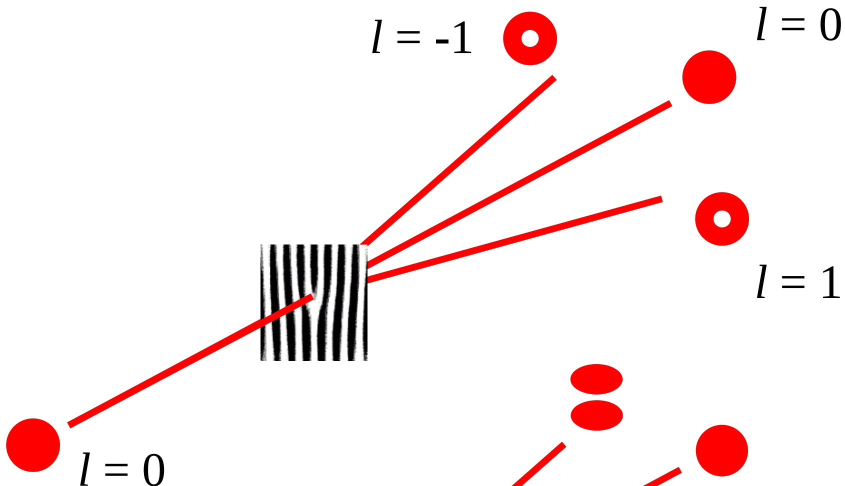


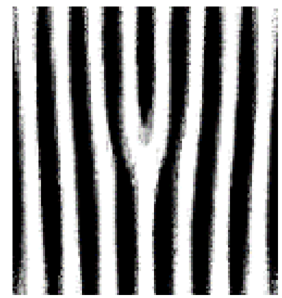
Holographic production of LG and HG beams

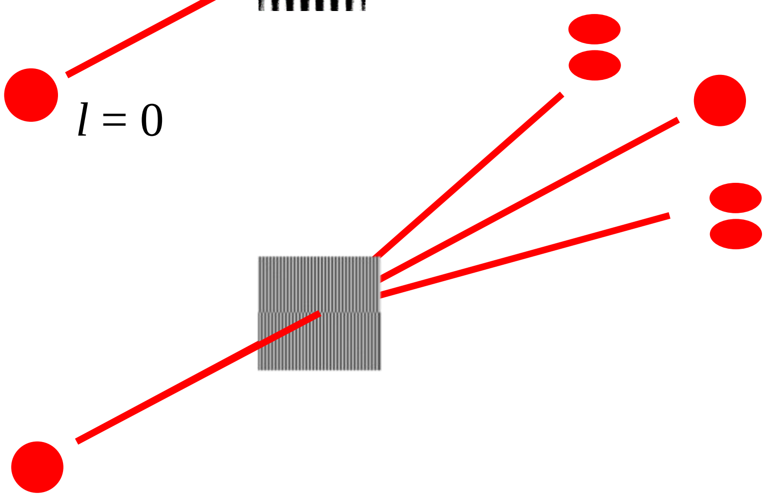


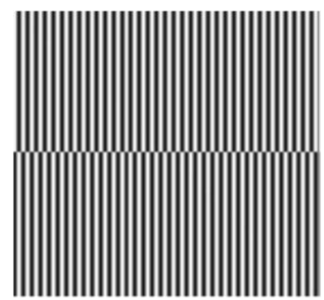


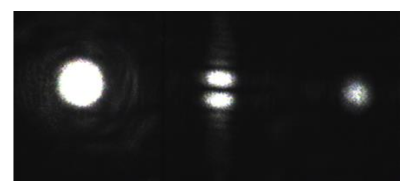
N.R. Heckenberg et al, Opt. Lett. **17**, 221 (1992)











Algebraic structure of paraxial wave functions

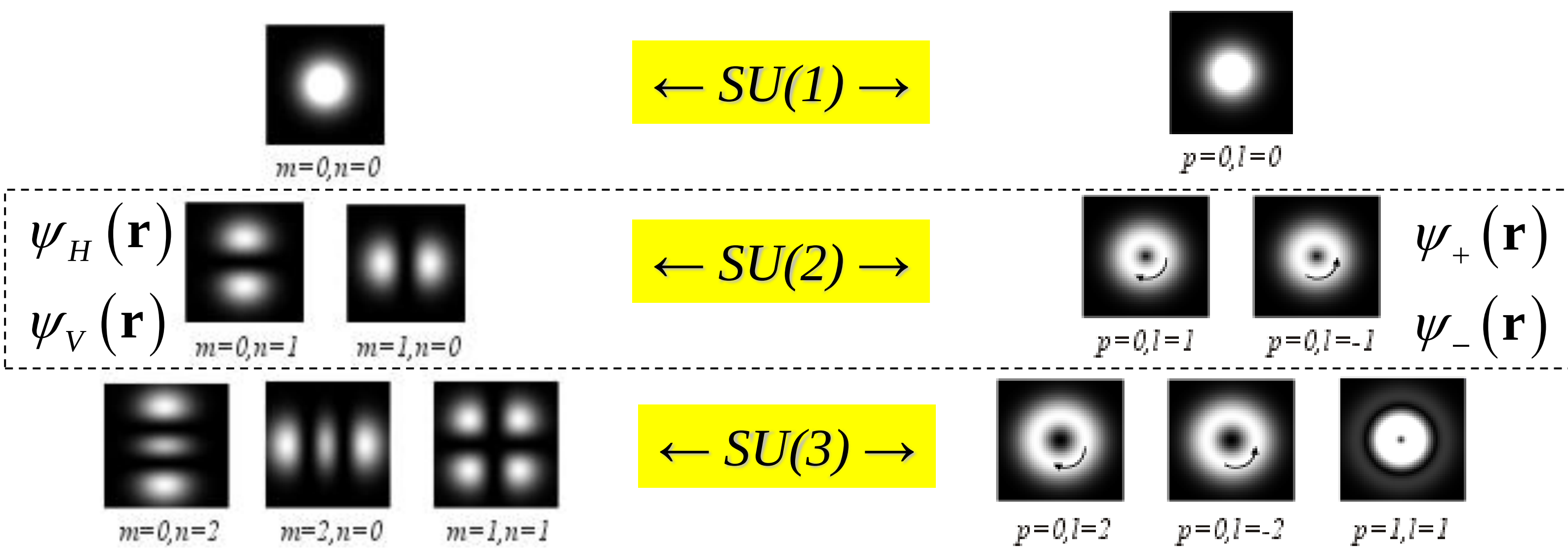
$$LG_{pl}(\mathbf{r}) = \sum_{k=0}^{N} \alpha_k \, HG_{k,N-k}(\mathbf{r})$$

E. Abramochkin and V. Volostnikov,
Opt. Commun. 83, 123 (1991)

LG-HG Unitary transformation

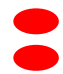
$$U_{nm}^{pl} = \sum_{\oplus} U^{(N)}$$

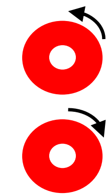
Hermite-Gauss (HG) **Laguerre-Gauss (LG)**

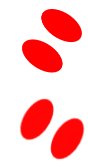


Astigmatic mode transformations

$$\psi_H(\mathbf{r}) \propto x\, e^{-(x^2+y^2)/w^2}$$ 

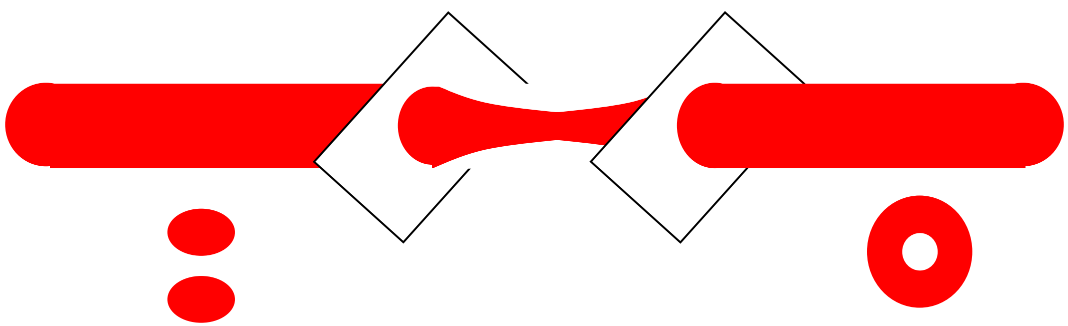
$$\psi_V(\mathbf{r}) \propto y\, e^{-(x^2+y^2)/w^2}$$ 

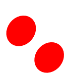
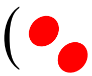
HG-LG $$\psi_{\pm} = \frac{\psi_H \pm i\,\psi_V}{\sqrt{2}}$$ 

HG-HG $$\psi_{\pm 45^\circ} = \frac{\psi_H \pm \psi_V}{\sqrt{2}}$$ 

Mode Converter

cylindrical lenses at 45º

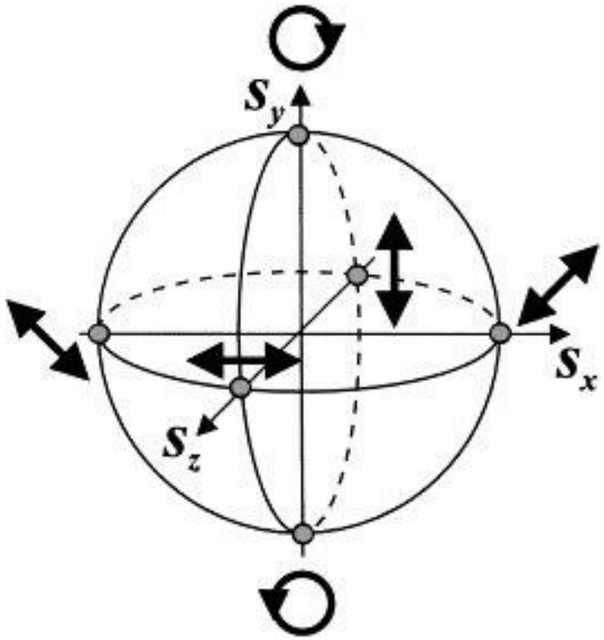


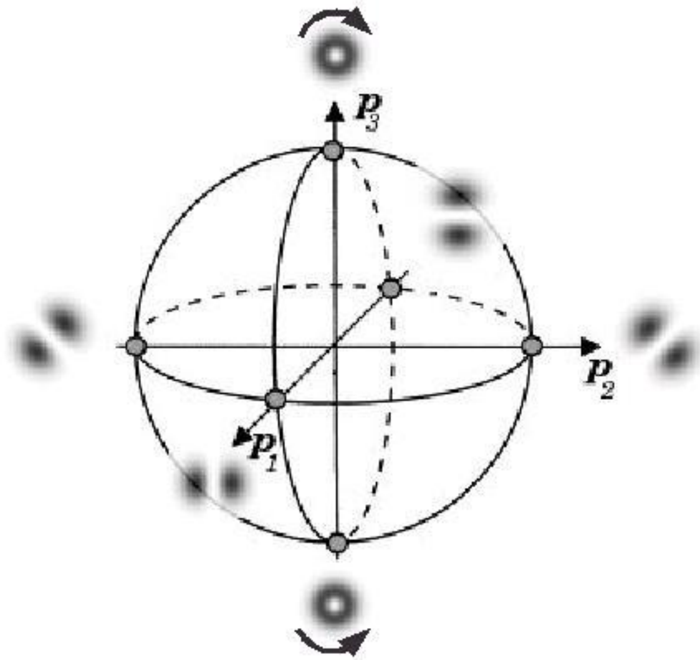
$\frac{1}{\sqrt{2}}($  $+$  $)$ → $\frac{1}{\sqrt{2}}($  $+\,i$  $)$

MC eigenvectors

$$\psi_{\pm 45^\circ} = \frac{\psi_H \pm \psi_V}{\sqrt{2}}$$ 

Poincaré representation





Poincaré sphere for polarization modes

Poincaré sphere for first order modes

$$\hat{\mathbf{e}}_{\theta,\varphi} = \cos\frac{\theta}{2}\,\hat{\mathbf{e}}_H + e^{i\varphi}\sin\frac{\theta}{2}\,\hat{\mathbf{e}}_V \qquad \psi_{\theta,\varphi} = \cos\frac{\theta}{2}\,\psi_H + e^{i\varphi}\sin\frac{\theta}{2}\,\psi_V$$